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SEQUENCES BY NUMBER OF *w*-RISES

BY

MORTON ABRAMSON

An m-permutation of n, repetitions allowed, is an m-sequence

$$e_1, e_2, \ldots, e_m \qquad e_i \in \{1, 2, \ldots, n\}. \tag{1}$$

A *w-rise* is a pair (e_i, e_{i+1}) such that $e_{i+1}-e_i \geq w > 0$. In this note we find an expression for $T_{k,w}(n, m)$, the number of m-sequences having precisely k w-rises. The case $w=1$ is given in [1] [2]. Also, when $w=1$ we give the number when each of the integers $1, 2, \ldots, r$ must appear at least once. Throughout we take $\binom{n}{k}=0$ for $n<0$ except where noted in (6).

In a sequence (1), let $P(i)$ be the property that $e_{i+1}-e_i \geq w$. There are a total of $m-1$ such properties. For each subset S of $Z_{m-1}=\{1, 2, \ldots, m-1\}$, let $A(S)$ be the number of sequences which have all of the properties $P(i)$ for $i \in S$ (and possibly others), and let

$$s(r) = \sum A(S) \tag{2}$$

where the summation extends over all subsets of order r. Thus, by the principle of inclusion and exclusion we find that

$$T_{k,w}(n, m) = \sum_{i=0}^{m-1-k} (-1)^i \binom{k+i}{k} s(k+i) \qquad (0 \leq k \leq m-1; 1 \leq w \leq n-1). \tag{3}$$

Hence, in order to evaluate $T_{k,w}(n, m)$, it is only necessary to evaluate $A(S)$.

Let S be a subset of Z_{m-1} of order r. We associate a composition $(b_1, \ldots, b_{m-r})$ of m with S as follows. If $i \in S$, we place i and $i+1$ in the same subset of Z_m; otherwise i is in a subset by itself. This obviously gives a set partition $B_1 \cup \cdots \cup B_{m-r}$ of Z_m and we set $b_j=|B_j|$ $(1 \leq j \leq m-r)$. For example, if $m=9$ and $S=\{2, 5, 6\}$, then we have $\{1\} \cup \{2, 3\} \cup \{4\} \cup \{5, 6, 7\} \cup \{8\} \cup \{9\}$ and $1+2+1+3+1+1=9$. Now, the sequence of elements indexed by elements of a subset B_j must form an incerasing sequence whose terms differ by at least w. Since there are [4, expression (23)]

$$\binom{n-(k-1)(w-1)}{k}$$

sequences $1 \leq x_1 < x_2 < \cdots < x_k \leq n$ which satisfy $x_{i+1}-x_i \geq w$ (such a sequence is equivalent to a sequence of k1's and $n-k$ 0's with every pair of 1's separated

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by at least $w-1$ 0's), we have shown that

$$A(S)=\prod_{j=1}^{m-r}\binom{n-(b_j-1)(w-1)}{b_j}. \tag{4}$$

Since the composition $(b_1, b_2, \dots, b_{m-r})$ determines the set S, we get by (2) and (4),

$$s(r)=\sum_{\substack{b_1+\cdots+b_{m-r}=m\\ b_j\geq 1}}\binom{n-(w-1)(b_1-1)}{b_1} \tag{5}$$

$$\times\binom{n-(w-1)(b_2-1)}{b_2}\cdots\binom{n-(w-1)(b_{m-r}-1)}{b_{m-r}}.$$

For the case $w=1$, expression (5) is simplified by using the generating function

$$\sum_m\sum_{\substack{b_1+\cdots+b_p=m\\ b_i>0}}\binom{n}{b_1}\cdots\binom{n}{b_p}x^m=[(1+x)^n-1]^p$$

$$=\sum_m\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\binom{nj}{m}x^m.$$

Simplifying in (3) $(0\leq k\leq m-1)$,

$$T_{k,1}(n,m)=\sum_{j=1}^{m-k}(-1)^{m-k-j}\binom{nj}{m}\sum_{i=0}^{m-k-j}\binom{k+i}{k}\binom{m-k-i}{j} \tag{6}$$

$$=\sum_{j=1}^{m-k}(-1)^{m-k-j}\binom{nj}{m}\binom{m+1}{m-k-j}$$

$$=\sum_{i=0}^{m-k-1}(-1)^i\binom{m+1}{i}\binom{n(m-k-i)}{m}$$

$\left(\text{and using [3, identity (3.150)] and } \binom{-n}{k}=(-1)^k\binom{n+k-1}{k}\right)$

$$=-\sum_{i=m-k}^{m+1}(-1)^i\binom{m+1}{i}\binom{n(m-k-i)}{m}$$

$$=\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}\binom{n(k+1-j)+m-1}{m},$$

[1, p. 356, e_1 is counted as an initial rise] and [2, p. 1091].

We now give the number of sequences (1) with precisely k 1-rises and with each of the integers $1, 2, \dots, r$ appearing at least once. These are the sequences counted by $T_{k,1}(n, m)$ but satisfying none of the properties, "integer i does not appear", $i=1, 2, \dots, r$. Hence, using the sieve formula the number is

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}T_{k,1}(n-i, m)\qquad (1\leq r\leq n)$$

$$=\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{m-1+(n-i)(k+1-j)}{m}.$$

In the case $m=r$, using [3, identity (3.150)], we obtain the familiar Eulerian number

$$\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}(k+1-j)^m,$$

which counts the number of permutations of $1, 2, \ldots, m$ with precisely k rises [5, pp. 216–19].

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COMMUTING k-TUPLES OF SELF ADJOINT OPERATORS AND MATRIX MEASURES

BY
PATRICK J. BROWNE

1. **Introduction.** In this note we give a generalization of a spectral representation theorem for self adjoint operators in a Hilbert space recently obtained by the author [1]. Our interest here is to develop corresponding results for a k-tuple $(T_1, \ldots, T_k)$ of commuting self adjoint operators.

In this section we present some definitions and terminology.

Definition 1. Let $\{\mu_{ij}\}$, $1 \le i, j \le n$, be a family of complex valued set functions defined on the bounded Borel subsets of R^k. The family (μ_{ij}) will be called an $n \times n$ positive matrix measure if

(i) the matrix $(\mu_{ij}(M))$ is Hermitian positive semi-definite for each bounded Borel set $M \subset R^k$;

(ii) we have

$$\mu_{ij}\left(\bigcup_{m=1}^{\infty} M_m\right) = \sum_{m=1}^{\infty} \mu_{ij}(M_m), \qquad 1 \le i, j \le n,$$

for each sequence (M_m) of disjoint Borel sets with bounded union.

The theory of matrix measures defined on the bounded Borel sets of R^1 is given in detail in [2, pp. 1337–1346]. A careful study of these pages shows that all properties of and results concerning matrix measures on R^1 carry over to the R^k case. We shall have occasion to call on these results in the R^k setting.

Definition 2. Let H be a Hilbert space and

$$T_i = \int_{-\infty}^{\infty} \lambda E_i(d\lambda), \qquad i = 1, 2, \ldots, k,$$

self adjoint operators in H. We say that the k-tuple $(T_1, \ldots, T_k)$ is a commuting k-tuple of self adjoint operators if for any choice of Borel sets $M_i \subset R$ we have

$$E_i(M_i)E_j(M_j) = E_j(M_j)E_i(M_i), \qquad 1 \le i, j \le k.$$

This definition follows Prugovečki [3, page 261] and we shall be leaning heavily on the theory of functions of such operators as outlined in the above reference (pp. 269–284).

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Specifically if E denotes the spectral measure on the Borel sets of R^k satisfying

$$E(M_1\times M_2\times\cdots\times M_k)=E_1(M_1)E_2(M_2)\cdots E_k(M_k)$$

whenever $M_i\subset R$, $i=1,\ldots,k$ are Borel sets and if $p(\lambda_1,\ldots,\lambda_k)$ is a Borel function on R^k, there exists a unique linear operator $p(T_1,\ldots,T_k)$ in H satisfying

$$(1)\qquad (p(T_1,\ldots,T_k)f,g)=\int_{R^k}p(\lambda)(E(d\lambda)f,g),\qquad g\in H.$$

Its domain of definition is given by

$$D(p(T_1,\ldots,T_k))=\left\{f\in H \;\middle|\; \int_{R^k}|p(\lambda)|^2\,(E(d\lambda)f,f)<\infty\right\}.$$

Details of these results can be found in [3, pp. 269–284]. Our definition of $p(T_1,\ldots,T_k)$ differs slightly from that of Prugovečki in that given a Borel function $p(\lambda)$ he establishes the existence of a linear operator $p(T_1,\ldots,T_k)$ such that

$$(2)\qquad (g,p(T_1,\ldots,T_k)f)=\int_{R^k}p(\lambda)(g,E(d\lambda)f),\qquad g\in H.$$

Our definition can be obtained by replacing $p(\lambda)$ by $\overline{p(\lambda)}$ in (2). For consistency with our earlier work [1] we shall use definition (1) which also ensures that

$$(\alpha_1p_1+\alpha_2p_2)(T_1,\ldots,T_k)=\alpha_1p_1(T_1,\ldots,T_k)+\alpha_2p_2(T_1,\ldots,T_k)$$

on the intersection of the domains of $p_1(T_1,\ldots,T_k)$ and $p_2(T_1,\ldots,T_k)$. Furthermore we shall restrict our attention to Borel functions p which are defined E almost everywhere on R^k. Under this condition it is easy to show that $D(p(T_1,\ldots,T_k))$ is dense in H, c.f. the corresponding result for a single self adjoint operator [2, Theorem 6, p. 1196]. The same argument also shows that $p(T_1,\ldots,T_k)$ is a closed operator in H.

2. **The spectral representation theorem.** Let H be a Hilbert space and $T_i=\int_{-\infty}^{\infty}\lambda E_i(d\lambda)$, $i=1,2,\ldots,k$, a commuting k-tuple of self adjoint operators in H. Denote by E, the spectral measure defined on Borel subsets of R^k and satisfying

$$E(M_1\times\cdots\times M_k)=E_1(M_1)\cdots E_k(M_k)$$

for all $M_1,\ldots,M_k$, Borel subsets of R^1.

For a finite set $N\subset H$, $N=\{f_1,\ldots,f_n\}$, H_N will denote the closure of the manifold in H consisting of all vectors of the form $p_1(T_1,\ldots,T_k)f_1+\cdots+p_n(T_1,\ldots,T_k)f_n$ where p_i varies over Borel functions defined E-almost everywhere on R^k for which $f_i\in D(p_i(T_1,\ldots,T_k))$, $1\leq i\leq n$.

The following Lemmas, Definition and Theorems have precise analogues in our earlier work dealing with a single self adjoint operator [1]. It is not our purpose here to reproduce verbatim existing arguments but rather to note that they carry over mutatis mutandis to the present situation. Accordingly, we take the liberty of stating these results without proof, referring the reader to the above reference.

Lemma 1. *Let the complex valued measures* μ_{ij}, $1 \leq i, j \leq n$, *be defined by* $\mu_{ij}(\cdot) = (E(\cdot)f_i, f_j)$. *Then* (μ_{ij}) *is an* $n \times n$ *positive matrix measure on the Borel subsets of* R^k.

Lemma 2. *Let* (μ_{ij}) *be as above. Define* μ *by* $\mu(\cdot) = \sum_{i=1}^n \mu_{ii}(\cdot)$. *Then* μ *is a totally finite regular measure on the Borel subsets of* R^k *and each* μ_{ij} *is absolutely continuous with respect to* μ. *Further,* H_N *is a Hilbert space which is unitarily equivalent to* $L^2(\mu_{ij})$.

Lemma 3. *Let* $n \geq 1$ *be an integer. Then there is a collection* $\mathscr{L}$ *of finite subsets* $N \subset H$, *each containing n points such that*

$$H = \sum_{N \in \mathscr{L}} \oplus H_N.$$

For $f \in H$ we let f_N be its component in H_N, that is

$$f = \sum_{N \in \mathscr{L}} f_N, f_N \in H_N.$$

Lemma 4. *For every Borel function p we have*

$$D(p(T_1, \ldots, T_k)) = \{f \in H \,|\, f_N \in D(p(T_1, \ldots, T_k)), N \in \mathscr{L}; \sum_{N \in \mathscr{L}} \|p(T_1, \ldots, T_k)f_N\|^2 \leq \infty\}$$

and $(p(T_1, \ldots, T_k)f)_N = p(T_1, \ldots, T_k)f_N$.

We now adopt the following notation:

$$N = \{f_1^N, f_2^N, \ldots, f_n^N\}, \mu_{ij}^N(\cdot) = (E(\cdot)f_i^N, f_j^N),$$

$$\mu_N = \sum_{i=1}^n \mu_{ii}^N, \mu_{ij}^N(M) = \int_M m_{ij}^N(t)\mu_N(dt).$$

Lemma 5. *H is unitarily equivalent to* $\sum_{N \in \mathscr{L}} L^2(\mu_{ij}^N)$.

Lemma 6. *For every Borel function k we have*

$$U\,D(p(T_1, \ldots, T_k)) = \left\{F \in \sum_N L^2(\mu_{ij}^N) \,\middle|\, \sum_N \int_{R^k} |p(t)|^2 \sum_{i,j=1}^n m_{ij}^N(t) F_{Ni}(t) \overline{F_{Nj}(t)} \mu_N(dt) < \infty \right\}$$

and

$$(U p(T_1, \ldots, T_k) f)_N(t) = p(t)(Uf)_N(t).$$

Here U denotes the unitary map $U: H \to \sum_N L^2(\mu_{ij}^N)$ established in the previous lemma. We have also used the notation for $F = \sum_N F_N \in \sum_N L^2(\mu_{ij}^N)$, $F_N = (F_{N1}, \ldots, F_{Nn})$.

Definition 3. Let $(T_1, \ldots, T_k)$ be a commuting k-tuple of self adjoint operators on a Hilbert space H and let $\{(\mu_{ij}^N)\}$ be a family of finite $n \times n$ positive matrix measures on the Borel subsets of R^k and vanishing on the complement of the Cartesian product of the spectra of the operators $T_1, \ldots, T_k$. Let U be a unitary map between H and $\sum_N L^2(\mu_{ij}^N)$. The transformation U is an $n\times n$ spectral representation of H onto $\sum_N L^2(\mu_{ij}^N)$ relative to $(T_1, \ldots, T_k)$ if the following conditions are satisfied:

(a) for every Borel function p defined E-almost everywhere on R^k we have

$$U\,D(p(T_1, \ldots, T_k)) = \left\{F \in \sum_N L^2(\mu_{ij}^N) \mid \sum_N \int_{R^k} |p(t)|^2 \sum_{i,j=1}^n F_{Ni}(t)\overline{F_{Nj}(t)}\mu_{ij}^N(dt) < \infty\right\};$$

(b) $(Up(T_1, \ldots, T_k)f)_N(t) = p(t)(Uf)_N(t)$.

THEOREM 1. *Every Hilbert space admits an $n\times n$ spectral representation relative to an arbitrary commuting k-tuple of self adjoint operators defined in it, for any value of the integer n.*

THEOREM 2. *Let U be an $n\times n$ spectral representation of H onto $\sum_K L^2(\rho_{ij}^K)$ relative to a commuting k-tuple of self adjoint operators $(T_1, \ldots, T_k)$. Then to each K there corresponds a finite set $N\subset H$ such that $(\rho_{ij}^K) = (\mu_{ij}^N)$, H is a direct sum of subspaces H_N and U maps H_N onto $L^2(\mu_{ij}^N)$.*

3. **An Analytic Representation.** We now specialize our attention to the case $H = L^2(S, \Sigma, \nu)$ where (S, Σ, ν) is a positive measure space. We shall further assume that there is an expanding sequence of sets $S_n \subset S$, covering S, each of which has finite ν measure and such that for bounded Borel sets $M\subset R^k$, the range of $E(M)$ contains only functions which are ν-essentially bounded on each of the sets S_n, (c.f. [2], p. 1210).

This restriction on the operators $T_1, \ldots, T_k$ will be assumed to hold throughout this section.

LEMMA 7. *Under the above hypothesis there is for each $g \in L^2(S, \Sigma, \nu)$ a function W defined on the Cartesian product of S and R^k which is measurable with respect to the product of ν and the measure $\mu = (E(\cdot)g, g)$ and which has the property that for every bounded Borel set $M\subset R^k$ and every F in $L^2(\mu)$ we have*

$$(i)\ \ \nu\text{-ess}\sup_{s\in S_n} \int_M |W(s, \lambda)|^2 \mu(d\lambda) < \infty,$$

$$(ii)\ \ (E(M)F(T_1, \ldots, T_k)g)(s) = \int_M W(s, \lambda)F(\lambda)\mu(d\lambda).$$

Proof. Again we are in the situation of being able to refer to an existing result proved for the case of a single operator noticing that the argument used there is applicable to our present case. We refer the reader to [2, Lemma 9, p. 1211].

In like manner, we follow the argument of Dunford and Schwartz and with reference to [2, Theorem 11, p. 1213] claim

THEOREM 3. *Let* (S, Σ, ν) *be a positive measure space and let* $\{S_n\}$ *be an increasing sequence of sets of finite measure covering* S. *Let* U *be a* 1×1 *spectral representation of* $L^2(S, \Sigma, \nu)$ *onto* $\sum_{N \in \mathcal{L}} L^2(\mu^N)$ *relative to the commuting k-tuple of self adjoint operators* $(T_1, \ldots, T_k)$. *Suppose for each bounded Borel set* $M \subset R^k$, *the range of* $E(M)$ *contains only functions which are* ν*-essentially bounded on each of the sets* S_n. *Then for each element* $N \in \mathcal{L}$ *there is a function* W_N *defined on* $S \times R^k$ *and having the properties:*

(a) W_N *is measurable with respect to the product measure* $\nu \times \mu^N$;

(b) *for each bounded Borel set* $M \subset R^k$ *we have*

$$\nu\text{-ess}\sup_{s \in S_n} \int_M |W_N(s, \lambda)|^2 \mu^N(d\lambda) < \infty, \qquad n \geq 1;$$

(c) $$(Uf)_N(\lambda) = \int_S f(s)\overline{W_N(s, \lambda)}\nu(ds), \qquad f \in L^2(S, \Sigma, \nu),$$

the integral existing in the mean square sense in $L^2(\mu^N)$.

This theorem gives an analytic representation of the spectral representation U as a collection of integral transforms in the case of a 1×1 spectral representation. The construction of an analytic representation for an $n \times n$ spectral representation is identical with that used for the theory of a single self adjoint operator. We refer the reader to [1, Lemmas 7, 8, Theorems 3, 4]. Accordingly we claim

THEOREM 4. *Let* U *be an* $n \times n$ *spectral representation of* $L^2(S, \Sigma, \nu)$ *onto* $\sum_{N \in \mathcal{L}} L^2(\mu_{ij}^N)$ *relative to the commuting k-tuple of self adjoint operators* $(T_1, \ldots, T_k)$. *Under the hypothesis of the previous theorem, for each* $N \in \mathcal{L}$ *there are functions* W_N^i, $1 \leq i \leq n$, *defined on* $S \times R^k$ *such that*

(a) W_N^i *is measurable with respect to* $\nu \times \mu_N$;

(b) *for each bounded Borel set* $M \subset R^k$ *we have*

$$\nu\text{-ess}\sup_{s \in S_n} \int_M \sum_{i,j=1}^n m_{ij}^N(t) W_N^i(s, t)\overline{W_N^j(s, t)}\mu_N(dt) < \infty, \qquad n \geq 1;$$

(c) $$((Uf)_N(t)) = \left(\int_S f(s)\overline{W_N^i(s, t)}\nu(ds)\right)_{i=1}^n, \qquad f \in L^2(S, \Sigma, \nu),$$

the integrals existing in the mean square sense in $L^2(\mu_{ij}^N)$.

THEOREM 5. *With the notation of the previous theorem we have*

$$f(s) = \sum_{N \in L} \int_{R^k} \sum_{i,j=1}^n m_{ij}^N(t)(Uf_N)_i(t) W_N^i(s, t)\mu_N(dt),$$

$f \in L^2(S, \Sigma, \nu)$, *the integrals existing in the mean square sense in* $L^2(S, \Sigma, \nu)$ *and the series converging in the norm of* $L^2(S, \Sigma, \nu)$.

Our project is now complete. The reader will see that by reference to existing results and noting their immediate generalization we have been able to produce a spectral representation theory for commuting k-tuples of self adjoint operators.

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EP OPERATORS AND GENERALIZED INVERSES

BY

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ABSTRACT: The relationship between properties of the generalized inverse of A, $A\dagger$, and of the adjoint of A, A^*, are studied. The property that $A\dagger A$ and $AA\dagger$ commute, called (E4), is investigated. (E4) generalizes the property of A being EP_r. A canonical form and a formula for $A\dagger$ are given if a matrix A is (E4). Results are in a Hilbert space setting whenever possible. Examples are given.

1. A bounded linear operator A on a complex Hilbert space $\mathscr{h}$ is called an EP operator if its range, $R(A)$, is closed and $R(A)=R(A^*)$. This concept was introduced for matrices by Schwerdtfeger in [14] and has been studied in detail by several authors [1], [9], [10], [11], [12], *et al.* For bounded linear operators A with closed range, the generalized inverse of A, $A^\dagger$, is defined to be the bounded operator $A_1^{-1}P$ where A_1 is the restriction of A to $R(A^*)$ and P is the orthogonal projection onto $R(A)$. Some equivalent definitions and properties of $A^\dagger$ are given in [2], [5], and [13].

There are some interesting relationships between normal operators, EP operators, and generalized inverses. One phenomenon which frequently occurs is that if one obtains a statement which characterizes a normal operator and replaces the adjoint operation (*) by the generalized inverse operation (†), then the resulting statement, which shall be referred to as the dual statement, is a characterization for EP operators. For example, consider the statement "A is a normal operator if and only if $A^*A=AA^*$." It is easy to verify that the dual statement "A is an EP operator if and only if $A^\dagger A=AA^\dagger$" is valid. Other relations of this type may be found in [10] and [12] where finite complex matrices were considered.

Our purpose in this paper is to explore further relationships between EP operators, generalized inverses, normal operators, and binormal operators. In particular, we completely characterize operators for which $A^\dagger A$ and $AA^\dagger$ commute by giving a block decomposition.

2. We begin by establishing some of the properties of (*) that we wish to exploit. All operators are bounded linear operators with closed ranges. Except possibly for those occurring as blocks in block matrices, they map a given Hilbert space into itself. $B(\mathscr{h})$ is the set of all bounded linear operators on $\mathscr{h}$ with closed range. If $X, Y\in B(\mathscr{h})$, then $[X, Y]=XY-YX$.

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THEOREM 1. *Suppose that $A \in B(\mathscr{h})$. Then each of the following implies the next.*
(N1) A is normal.
(N2) A^2 is normal.
(N3) $[A^*A, AA^*]=0$.

Proof. (N1)⇒(N2) is clear. To show that (N2)⇒(N3), assume that A^2 is normal. Fuglede's Theorem asserts that if B is normal and $[C, B]=0$, then $[C^*, B]=0$. (See [8, pp. 88–89] for a discussion of Fuglede's Theorem). Thus $[A^*, A^2]=0$ and $[A, A^{*2}]=0$ since $[A, A^2]=0$ and $[A^*, A^{*2}]=0$. (N3) now follows since

$$(AA^*)(A^*A) = AA^{*2}A = A^{*2}A^2 = A^*A^2A^* = (A^*A)(AA^*).$$

Operators satisfying (N3) were studied in [4] where they were referred to as binormal operators. (N3) also appears in the work of Embry [7]. Examples exist to show that none of the implications in Theorem 1 reverse. Condition (N2) may be replaced by

(N4) $[A, A^*A]=0$.

Operators satisfying (N4) have been studied by Brown [3]. Implicitly contained in [3] is the fact that if the underlying Hilbert space is finite dimensional, then (N4) is equivalent to (N1).

3. The next result is the (†) analogue of Theorem 1. It is interesting that some of the same examples used to show the implications are irreversible will work for Theorem 2 as for Theorem 1.

THEOREM 2. *Suppose that $A \in B(\mathscr{h})$. Then each of the following implies the next.*
(N1) *A is normal.*
(E1) *A is* EP.
(E2) *A^2 is* EP.
(E3) $(A^\dagger)^2=(A^2)^\dagger$.
(E4) $[A^\dagger A, AA^\dagger]=0$.
Furthermore all the implications are proper.

Proof. (N1)⇒(E1) is well known. That (E1)⇒(E2) is also clear. If $B, C \in B(\mathscr{h})$ and $BCB=B$, then C is called a (1)-inverse for B. If $BCB=B$ and $CBC=C$, then C is called a (1, 2)-inverse for B. Clearly $B^\dagger$ is a (1, 2)-inverse for B. We will show later in Theorem 3 that (E4) is equivalent to $A^{\dagger 2}$ being a (1) or (1, 2)-inverse for A^2. Then (E3) will imply (E4). We will now show (E2)⇒(E3). Suppose that A^2 is EP. Let $N(A^2)$ denote the null space of A^2. Then relative to the orthogonal decomposition $\mathscr{h}=R(A^2)\oplus N(A^2)$, we have $A^2=\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ where T is invertible. Let $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$. Then $[A, A^2]=0$ and the invertibility of T imply that $C=0$ and $D=0$. Hence $E^2=0$. But $A^\dagger=\begin{bmatrix} B^{-1} & 0 \\ 0 & E^\dagger \end{bmatrix}$ and $E^{\dagger 2}=0$ so that $A^{\dagger 2}=A^{2\dagger}$.

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The same proof shows that if A^n is EP, then $A^{\dagger n}=A^{n\dagger}$.

Example 1. Let $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$. Then A^2 is EP while A is not. Thus (E2)$\not\Rightarrow$(E1).

Example 2. Let

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

Then $A^{\dagger}=A^*$ and $A^{2\dagger}=A^{2*}$. Hence $A^{2\dagger}=A^{2*}=A^{*2}=A^{2\dagger}$. But $R(A^2)$ and $R(A^{*2})$ are unequal and hence A^2 is not EP. Thus (E3)$\not\Rightarrow$(E2).

Example 3. Let

$$A=\begin{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}&0&\begin{bmatrix}1&0\\0&1\end{bmatrix}\\\begin{bmatrix}0&0\\1&0\end{bmatrix}&0&\begin{bmatrix}1&0\\0&1\end{bmatrix}\\0&0&0\end{bmatrix}\quad\text{where}\quad 0=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Then

$$A^{\dagger}=\begin{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}&\begin{bmatrix}0&1\\-1&0\end{bmatrix}&0\\0&0&0\\\begin{bmatrix}0&0\\0&1\end{bmatrix}&\begin{bmatrix}1&0\\0&0\end{bmatrix}&0\end{bmatrix}.$$

A direct calculation shows that $A^2A^{\dagger 2}$ is not self-adjoint. Hence $A^{\dagger 2}\neq A^{2\dagger}$. However, it can be verified that $[AA^{\dagger}, A^{\dagger}A]=0$. Thus (E4)$\not\Rightarrow$(E3).

Before proceeding we need the following well-known lemma [15, p. 58].

Lemma 1. *If $P\in B(\mathscr{h})$, $P^2=P$, and $\|P\|\leq 1$, then P is a self-adjoint projection.*

One use of condition (E4) is the following.

Theorem 3. *Let $A\in B(\mathscr{h})$. Then $A^{\dagger 2}$ is a (1)-inverse for A^2 if and only if $[A^{\dagger}A, AA^{\dagger}]=0$.*

Proof. Suppose $A^{\dagger 2}$ is a (1)-inverse for A^2. Then $A^2A^{\dagger 2}A^2=A^2$. Multiplying on the right and left by $A^{\dagger}$ gives $(A^{\dagger}A^2A^{\dagger})(A^{\dagger}A^2A^{\dagger})=A^{\dagger}A^2A^{\dagger}$. But $\|A^{\dagger}A^2A^{\dagger}\|\leq\|A^{\dagger}A\|\,\|AA^{\dagger}\|=1$. Thus $(A^{\dagger}A)(AA^{\dagger})=A^{\dagger}A^2A^{\dagger}$ is self-adjoint by Lemma 1. Since $A^{\dagger}A$ and $AA^{\dagger}$ are self-adjoint, and their product is self-adjoint, we have $[A^{\dagger}A, AA^{\dagger}]=0$ as desired. Now suppose $A^{\dagger}AAA^{\dagger}=AA^{\dagger}A^{\dagger}A$. Multiplying on the right and left by A yields the desired result.

Note that if $A^{\dagger 2}$ is a (1)-inverse for A^2, then it is a (1, 2)-inverse. Thus if (E4) is known to hold one may calculate (1, 2)-inverses for A^2 from $A^{\dagger}$ with little additional work.

4. EP operators have the advantage of a simple canonical form, $\begin{bmatrix}T&0\\0&0\end{bmatrix}$, and an easily computed (†), $\begin{bmatrix}T^{-1}&0\\0&0\end{bmatrix}$. Unfortunately, even in the finite dimensional case, EP matrices form a very restrictive class. Not all matrices are even similar to an EP matrix.

As we will see shortly, operators satisfying (E4) also have a nice canonical form and a (†) that is fairly easy to compute. However, they admit a much greater degree of variety, for every matrix is similar to a matrix satisfying (E4). This can be easily seen by observing that every block in the Jordan form is (E4). In fact, the Jordan form can be written $\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ where J_1 is invertible, if it is present (hence (E1)), and J_2 is (E3).

5. A condition implying (E4) will now be established.

THEOREM 4. *Let* $A \in B(\mathscr{h})$. *If* $[A^*A, AA^*]=0$, *then* $[A^\dagger A, AA^\dagger]=0$.

Proof. Let $\sigma(A)$ be the spectrum of A. It is known that $\sigma(A^*A)$ and $\sigma(AA^*)$ have the same nonzero elements. Let f be the function defined on the real line by $f(0)=0$ and $f(\lambda)=1$ if $\lambda\neq 0$. Using the spectral theorem for self-adjoint operators as found in [6, Chapter X], we can define $f(C)$ for any self-adjoint operator C. Then $f(C)$ is a self-adjoint projection onto the closure of $R(C)$. Suppose that $[A^*A, AA^*]=0$. Then $[f(A^*A), f(AA^*)]=0$. But $R(A^*A)=R(A^*)$ and $R(AA^*)=R(A)$. Hence $A^\dagger A=f(A^*A)$, $AA^\dagger=f(AA^*)$, and the result follows.

Notice that $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ satisfies (E1), and hence (E4), but not (N3) so that the converse of Theorem 4 is not valid. In general, the (E) conditions are geometric statements about ranges, whereas the (N) conditions convey information on the entries of a matrix of A, that is, quantitative information. Thus one would not expect conditions of the (E) type to imply any of the (N) type without additional quantitative assumptions.

The next section is concerned primarily with the matrix case and will contain applications of some of our earlier ideas.

6. To determine whether A is EP it is necessary in principle, to determine $R(A)$ and $R(A^*)$ or calculate $A^\dagger$. In applications using matrices this can be a time consuming process if the matrices are large. To find $A^\dagger A$ or $AA^\dagger$ requires finding only one of the ranges. With the exception of parts 2 and 3, which are included for completeness, the next theorem contains several potentially useful reformulations of EP for matrices. The theorem is motivated by a result of Embry [7] which states that if both A^*A and AA^* commute with $A+A^*$, then A is normal.

THEOREM 5. *Suppose that A is an* $n\times n$ *complex matrix. Then the following conditions are equivalent.*

(1) A *is* EP.
(2) $[A^\dagger A, A+A^\dagger]=0$.
(3) $[AA^\dagger, A+A^\dagger]=0$.
(4) $[A^\dagger A, A+A^*]=0$.

(5) $[AA^\dagger, A+A^*]=0$.
(6) $[A, A^\dagger A]=0$.
(7) $[A, AA^\dagger]=0$.

Proof. Clearly (1) implies the rest. To show that (2)$\Rightarrow$(1), assume that $A^\dagger A(A+A^\dagger)=(A+A^\dagger)A^\dagger A$. Then $A^\dagger A^2+A^\dagger-A^{\dagger 2}A=A$. Hence $R(A)\subseteq R(A^\dagger)=R(A^*)$. Thus $R(A)=R(A^*)$ and A is EP. The proofs of (3)$\Rightarrow$(1), (4)$\Rightarrow$(1), and (5)$\Rightarrow$(1) are similar. Suppose then (6) so that $A(A^\dagger A)=(A^\dagger A)A$ or $A=A^\dagger A^2$.

Then $AA^\dagger=(A^\dagger A)(AA^\dagger)$. Since $A^\dagger A$ and $AA^\dagger$ are self-adjoint projections of the same rank, this implies $A^\dagger A=AA^\dagger$ and A is EP. The final implication (7)$\Rightarrow$(1) is similar to (6)$\Rightarrow$(1).

Example 4. Suppose $\{e_0, e_1, \ldots\}$ is an orthonormal basis for a separable, infinite dimensional, Hilbert space. Let S be the bilateral shift defined by $Se_i=e_{i+1}$ and extended linearly. Let P be the projection onto the subspace spanned by $\{e_1, e_2, \ldots\}$. It is easy to verify that $S^*=S^\dagger$ and $SS^*=P$ while $S^*S=I$, the identity Thus S is not EP since $SS^\dagger\neq S^\dagger S$. But S satisfies (2), (4), and (6). S^* is not EP and satisfies (3), (5), and (7). The assumption of finite dimensionality was thus crucial to all parts of Theorem 5.

The dual to Embry's result is valid even if H is infinite dimensional.

Theorem 6. *Let* $A\in B(\hbar)$. *If* $[A^\dagger A, A+A^\dagger]=0$ *and* $[AA^\dagger, A+A^\dagger]=0$, *then* A *is* EP.

Proof. Suppose $[A^\dagger A, A+A^\dagger]=0$. Then $A^\dagger A^2+A^\dagger=A+A^{\dagger 2}A$. Multiplication on the right by $A^\dagger$ gives $(A^\dagger A)(AA^\dagger)=AA^\dagger$. Thus $R(A)\subseteq R(A^*)$. Similarily $[AA^\dagger, A+A^\dagger]=0$ gives $R(A^*)\subseteq R(A)$ and hence A is EP.

In [10] it was shown for matrices, that if $R(A^2)=R(A)$ and $A^{\dagger 2}=A^{2\dagger}$, then A is EP. We now improve this result.

Theorem 7. *Let* A *be an* $n\times n$ *complex matrix. If* $R(A^2)=R(A)$ *and* $[A^\dagger A, AA^\dagger]=0$, *then* A *is* EP.

Proof. Suppose that $[A^\dagger A, AA^\dagger]=0$, that is, that the projections onto $R(A)$ and $N(A)$ commute. Thus $\mathbb{C}^n=R(A)\cap R(A^*)\oplus R(A)\cap N(A)\oplus N(A^*)\cap N(A)\oplus N(A^*)\cap R(A^*)$. But if $R(A^2)=R(A)$, then $R(A)\cap N(A)=\{0\}$. Since $\mathbb{C}^n$ is finite dimensional, we also have $R(A^{*2})=R(A^*)$ and hence $R(A^*)\cap N(A^*)=\{0\}$. Thus $N(A)$ is perpendicular to $R(A)$ and A is EP.

If $A=S^*$, S as in Example 4, then $R(A^2)=R(A)$ and $A^{2\dagger}=A^{\dagger 2}$ but A is not EP. Thus finite dimensionality was needed both for Theorem 7 and the original result in [10].

Not all the results of [10] can be improved by the substitution of one of our conditions. For example, it was shown in [10] that the matrix A is normal if and only if A is EP and $[A^\dagger, A^*]=0$. We cannot weaken A is EP to any of (E2), (E3), or

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(E4). $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ satisfies (E2), and $[A^\dagger, A^*]=0$, while A is not normal.

7. The condition $[A^*, A^\dagger]=0$ is different from our (E) conditions in the quantitative way discussed earlier. We give two examples to show that it is actually independent of (E4).

EXAMPLE 5. Let

$$A=\begin{bmatrix}0 & \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} & 0\\ 0 & 0 & 0\\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & 0 & 0\end{bmatrix}.$$

Then

$$A^\dagger=\begin{bmatrix}0 & 0 & \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{and} \quad [AA^\dagger, A^\dagger A]=0,$$

but $[A^\dagger, A^*]\neq 0$.

EXAMPLE 6. Let $A=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 0\end{bmatrix}$. Then $A^*=A^\dagger$ and hence $[A^*, A^\dagger]=0$. But $[A^\dagger A, AA^\dagger]\neq 0$. Notice also that $R(A^2)=R(A)$.

8. The proof of Theorem 7 suggests that matrices satisfying (E4) must have a nice standard form.

THEOREM 8. *If A is an $n\times n$ matrix and $[A^\dagger A, AA^\dagger]=0$, then there exists a unitary matrix U, and matrices A_{11}, A_{12}, A_{31}, A_{32} such that*

$$A=U\begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ 0 & 0 & 0 & 0\\ A_{31} & A_{32} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}U^*=U\begin{bmatrix}I & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & I\end{bmatrix}\begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{31} & A_{32} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}U^*$$

where $\begin{bmatrix}A_{11} & A_{12}\\ A_{31} & A_{32}\end{bmatrix}$ is invertible and each I is an identity matrix.

Proof. Suppose $[A^\dagger A, AA^\dagger]=0$. Consider A as a linear transformation on $\mathbb{C}^n$ with the standard basis. Since $[A^\dagger A, AA^\dagger]=0$ we have

$$\mathbb{C}^n=R(A)\cap R(A^*)\oplus R(A^*)\cap N(A^*)\oplus R(A)\cap N(A)\oplus N(A)\cap N(A^*).$$

Pick an orthonormal basis for each summand and combine to get an orthonormal basis for $\mathbb{C}^n$. Let U^*AU be the matrix of A relative to this new basis. Relative to the decomposition of $\mathbb{C}^n$ we have $U^*AU=[A_{ij}]$, $1\leq i, j\leq 4$, a 4×4 block matrix. An easy computation shows that $A_{ij}=0$ except possibly for A_{11}, A_{12}, A_{31}, and A_{32}. We note that $\dim[R(A^*)\cap N(A^*)]=\dim[R(A)\cap N(A)]$ since rank A^*-

rank A^{*2}=rank A−rank A^2. That $\begin{bmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$ is invertible follows from dim $R(A)$=dim $R(A^*)$ and the decomposition.

If (E4) is satisfied and A is an $n\times n$ matrix, then $A^\dagger$ can be calculated from U and $\begin{bmatrix} C_{11} & C_{13} \\ C_{21} & C_{23} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}$. In fact,

$$A^\dagger = U\begin{bmatrix} C_{11} & 0 & C_{13} & 0 \\ C_{21} & 0 & C_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*.$$

The standard form in Theorem 8 has several uses. In addition to providing a way of calculating $A^\dagger$, it can also be useful in producing examples and counter examples. The standard form was used in constructing Examples 3 and 5.

If $\mathscr{h}$ is allowed to be infinite dimensional, then we still get the first block form of Theorem 8, but $\begin{bmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$ may no longer be invertible or square.

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A COMPARISON BETWEEN THIN SETS AND GENERALIZED AZARIN SETS

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1. **Introduction.** Let $\mathbf{R}^p(p\geq 2)$ denote p-dimensional Euclidean space, D the half space defined by $\{P=(x_1, x_2, \ldots, x_p)\in \mathbf{R}^p : x_p>0\}$ and ∂D the frontier of D in $\mathbf{R}^p$. The Martin boundary (see [2]) of D can be identified with $\partial D \cup \{\infty\}$. We recall that the function $h(P)=x_p$ is a minimal harmonic function on D with pole at ∞. In 1949 (see [6]), Mme. Lelong defined thin sets at the boundary of D, including the point at ∞, of the type that we shall in future refer to as *minimally* thin sets. If u is a subharmonic function on D such that $\limsup u\leq 0$ everywhere on ∂D and $\sup u(P)/x_p=\beta<\infty$, she proved ([6], Théorème 1a) that there exists a set $E\subset D$, minimally thin at ∞, such that $\lim u(P)/x_p=\beta$, $P\to\infty$, $P\in D\backslash E$. Conversely (see [2] and [6], Reciproque 1a), her results imply that if $E\subset D$ is minimally thin at ∞, then there exists a subharmonic function u on D satisfying the above conditions such that

$$\limsup_{P\to\infty,\, P\in E} u(P)/x_p < \limsup_{P\to\infty,\, P\in D} u(P)/x_p = \beta.$$

If $P=(x_1, x_2, \ldots, x_p)$ and θ is defined such that $\cos\theta=x_p/|P|$, $0\leq\theta<\pi/2$, then Azarin [1] has obtained results which imply that if u is subharmonic on D and subject to the above restrictions, then there exists a set $F\subset D$ such that $\lim u(P)/|P|=\beta\cos\theta$, $P\to\infty$, $P\in D-F$, uniformly in θ where the exceptional set F satisfies the following thinness condition:

(1.1) There exists a sequence of balls $\{B_n\}$ which covers F such that $\sum_n (r_n/R_n)^{p-1}<\infty$, where r_n is the radius of B_n and R_n is the distance between its centre and the origin.

We note here that the results of Azarin have since been generalized by Essén and Lewis (see [4]).

If E is contained in a Stolz domain $K=\{P\in D: 0\leq\theta\leq\theta_0<\pi/2\}$, then $\lim((u(P)/x_p)-\beta)=0$, $P\to\infty$, $P\in K\backslash E$, if and only if $\lim((u(P)/|P|)-\beta\cos\theta)=0$, $P\to\infty$, $P\in K\backslash E$, uniformly in θ. If $E\subset K$ is a minimally thin set at ∞ with respect to D, it follows that E must also satisfy Azarin's condition (1.1). We shall show, however, that the converse does not hold (see in particular the corollaries of

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Theorem 5). Our main purpose here is to work out some of the relations between minimally thin sets at ∞ which are restricted to a Stolz domain, and those which satisfy a generalized form of Azarin's condition. We point out here that Azarin's work, which extends and unifies earlier generalizations of the Ahlfors-Heins version of the Phragmén-Lindelöf principle, is applicable to exceptional sets in the half space itself. We only claim to improve on the results of Azarin when the exceptional set in question is restricted to a Stolz domain.

DEFINITION 1. Let $h:[0, \infty)\to[0, \infty)$ be a continuous non-decreasing function such that $h(0)=0$. We define $E\subset D$ to be a generalized Azarin set with respect to h if and only if there exists a sequence of balls $\{B_n\}$ such that their union covers E and such that $\sum h(r_n/R_n)<\infty$, where r_n is the radius of B_n and R_n is the distance between its centre and the origin such that $R_n\to\infty$ as $n\to\infty$.

DEFINITION 2. If $\alpha>0$ and $h(r)=r^\alpha$, $r>0$, we shall say that E is a generalized Azarin set of order α if it is a generalized Azarin set with respect to h. In $\mathbf{R}^p$ $(p\geq 2)$, a generalized Azarin set of order $p-1$ shall simply be called an Azarin set (see (1.1)).

REMARK 1. Any generalized Azarin set of order α is also a generalized Azarin set of order α', whenever $\alpha'>\alpha$.

NOTATION:

(i) B_n means an open ball in $\mathbf{R}^p$ of radius r_n and centre P_n, where $|P_n|=R_n$

(ii) K is the Stolz domain $\{P\in D:0\leq 0\leq \theta_0<\pi/2\}$.

(iii) Let $s>1$ be fixed and I_n be the intersphere $\{P\in\mathbf{R}^p:s^n\leq|P|<s^{n+1}\}$.

(iv) $$\gamma(r)=\begin{cases}(\log(1/r))^{-1}, & 0<r<s^{-1},\\ (\log s)^{-1} & r\geq s^{-1}.\\ 0, & r=0.\end{cases}$$

(v) If $E\subset D$ is a given set, $c(E)$ denotes the ordinary (Newtonian if $p=3$) capacity in $\mathbf{R}^p$ of E and we define $E_n=E\cap I_n$ so that $c_n=c(E_n)$. In $\mathbf{R}^2$, $\lambda(E)$ denotes the logarithmic capacity of E and $\lambda_n=\lambda(E_n)$.

Let us also state known criteria for minimal thinness at ∞ for a set E contained in a Stolz domain: In $\mathbf{R}^p$, $p\geq 3$, a necessary and sufficient condition is that

$$\sum_{n=1}^{\infty} c_n/s^{n(p-2)}<\infty. \tag{1.2}$$

(see Mme. Lelong [6], p. 131). In $\mathbf{R}^2$, a necessary and sufficient condition is that

$$\sum_{n=1}^{\infty}\gamma(\lambda_n/s^n)<\infty. \tag{1.3}$$

(see Jackson [5], Theorem 1′).

2. **Some results on generalized Azarin sets.** When is a generalized Azarin set minimally thin at ∞? We start with two preliminary results.

THEOREM 1. *If* $E=\bigcup_{n=1}^{\infty} B_n$ *is a subset of* $\mathbf{R}^p (p\geq 3)$ *such that each ball* $B_n \subset I_n \cap K$, *then E is minimally thin at* ∞ *if and only if it is a generalized Azarin set of order* $p-2$.

Proof. Since E is contained in the Stolz domain K, we can use (1.2). $E\cap I_n=B_n$ is a ball of radius r_n, and it is clear that $c_n=r_n^{p-2}$. We also know that $s^n\leq R_n\leq s^{n+1}$. Hence $c_n/s^{n(p-2)}\geq (r_n/R_n)^{p-2}\geq c_n/s^{(n+1)(p-2)}$. Since the series $\sum c_n/s^{n(p-2)}$ and $\sum c_n/s^{(n+1)(p-2)}$ are co-convergent, the theorem is proved.

REMARK 2. When $p\geq 3$, it is clear that Azarin's condition (1.1) does not characterize minimally thin sets at ∞ which are restricted to a Stolz domain.

THEOREM 2. *If* $E=\bigcup_n B_n$ *is defined as in Theorem* 1 *subject to the modification that* $E\subset \mathbf{R}^2$, *then E is minimally thin at* ∞ *if and only if it is a generalized Azarin set with respect to* γ. *This means that*

$$\sum \gamma(r_n/R_n) < \infty. \tag{2.1}$$

Proof. E is minimally thin at ∞ if and only if (1.3) is true. Since $E\cap I_n=B_n$ is a disc of radius r_n, $\lambda_n=r_n$. Thus Theorem 2 will be proved if we can show that (2.1) is equivalent to

$$\sum \gamma(r_n/s^n) < \infty. \tag{2.2}$$

Since $s^n\leq R_n< s^{n+1}$ it is clear that $r_n/s^n\to 0$, $n\to\infty$, if either one of these two conditions hold. The function γ is nondecreasing and hence

$$\gamma(r_n/s^{n+1}) \leq \gamma(r_n/R_n) \leq \gamma(r_n/s^n). \tag{2.3}$$

Since

$$\gamma(xy) = \gamma(x)\gamma(y)(\gamma(x)+\gamma(y))^{-1}, \qquad x, y \in (0, s^{-1}],$$

we also know that if n is large enough,

$$\gamma(r_n/s^{n+1}) = \gamma(r_n/s^n)\gamma(1/s)(\gamma(r_n/s^n)+\gamma(1/s))^{-1}. \tag{2.4}$$

But $\gamma(1/s)>0$, and it is clear from (2.3) and (2.4) that conditions (2.1) and (2.2) are equivalent.

REMARK 3. Let E be constructed as in Theorem 2 so that $r_n/R_n=s^{-n}$ for all n. Then E is a generalized Azarin set of order α for all $\alpha>0$, but E is not minimally thin at ∞. We can conclude that in two dimensions a minimally thin set at ∞ which is restricted to a Stolz domain cannot be characterized as a generalized Azarin set of any order.

We shall now consider a sufficient condition for minimal thinness.

THEOREM 3. *If* $E\subset K\subset \mathbf{R}^p$, $(p\geq 3)$ *such that E is a generalized Azarin set of order* $p-2$, *then E is minimally thin at* ∞.

Proof. By assumption, there exists a sequence of balls $\{B_n\}$ which covers E such that $\sum (r_n/R_n)^{p-2}<\infty$. We now re-label the sequence $\{B_n\}$ to form a double sequence $\{B_{nj}\}$ such that the centre of each ball B_{nj} lies in the intersphere I_n. Hence $s^n\leq R_{nj}<s^{n+1}$, and it follows that $\sum_{n,j} (r_{nj}/R_{nj})^{p-2}<\infty$ if and only if $\sum_{n,j} (r_{nj}/s^n)^{p-2}<\infty$. Let c_{nj} denote the ordinary capacity in $\mathbf{R}^p$ ($p\geq 3$) of $E_{nj}=E\cap B_{nj}$. Then $c_{nj}\leq c(B_{nj})=r_{nj}^{p-2}$ so that

$$\sum_{n,j} c_{nj}s^{-n(p-2)}<\infty. \tag{2.5}$$

Since the capacity function is countably subadditive (see [2], chapters IV, VIII), $c(\bigcup_j E_{nj})\leq \sum_j c_{nj}$. Hence with the possible exception of at most finitely many n, we have

$$c_n(E)=c(E_n)\leq \sum_j (c_{n-1,j}+c_{nj}+c_{n+1,j}).$$

It then follows from (2.5) that $\sum c_n(E)s^{-n(p-2)}<\infty$. By (1.2), E is minimally thin at ∞, and Theorem 3 is proved.

Theorem 4. *If $E\subset K\subset \mathbf{R}^2$ is a generalized Azarin set with respect to γ, then E is minimally thin at ∞.*

Proof. Arguing as in the previous proof, we have a sequence of balls $\{B_{nj}\}$ such that $s^n\leq R_{nj}<s^{n+1}$ and such that $\sum_{n,j}\gamma(r_{nj}/s^n)<\infty$. If $E_{nj}=E\cap B_{nj}$, then $\lambda(E_{nj})\leq r_{nj}$. Furthermore, $\lambda(E_{nj}/s^n)=s^{-n}\lambda(E_{nj})$ (see [7], p. 56) and therefore $\sum_{n,j}\gamma\circ\lambda(E_{nj}/s^n)<\infty$. The ordinary capacity function $\gamma\circ\lambda$ is countably subadditive (even though λ fails to have this property). Therefore

$$\sum_{n=2}^{\infty}\gamma(\lambda_n(E)/s^n)=\sum_{n=2}^{\infty}\gamma\circ\lambda(E_n/s^n)\leq 3\sum_{n,j}\gamma\circ\lambda(E_{nj}/s^n)<\infty.$$

By applying (1.3) we conclude that E is minimally thin at ∞.

Remark 4. The results of this section remain valid for minimally thin sets at the origin provided that we modify our sequence of balls $\{B_n\}$ in such a way that the distance between the origin and each B_n is greater than zero.

Remark 5. If $p=3$, the axis which is normal to ∂D constitutes an example of a minimally thin set at ∞ which is not an Azarin set of order 1. The implication proved in Theorem 3 is therefore strict in general.

3. A Lemma.

In the sequel, we need certain results from Carleson ([3], §§II–IV). Let $h\colon [0,\infty)\to[0,\infty)$ be a continuous, non-decreasing function such that $h(0)=0$. If E is a bounded set, consider a countable number of balls $\{B_n\}$ with radii $\{r_n\}$ such that $E\subset \bigcup_n B_n$, and define $M_h(E)=\inf\sum h(r_n)$ for all such coverings. For every compact set F, there is a non-negative set function μ depending on F such that

$$\mu(B)\leq h(r) \tag{3.1}$$

for every ball B of radius r, and such that the following inequality holds:

$$M_h(F) \leq \text{const.}\, \mu(F). \tag{3.2}$$

The constant depends only on the dimension p (see [3], Theorem 1 p. 7).

We want to study the relation between the outer measure M_h and outer capacity C_L with respect to a kernel L which is defined in the following way.

Let us define

$$\phi_p(r) = \begin{cases} \log(1/r), & p = 2, r > 0. \\ r^{2-p} \quad , & p > 2, r > 0. \end{cases}$$

If H is a non-negative, continuous increasing function on R, we consider kernels of the form $L=H \circ \phi_p$ which are restricted so that

$$\int_0 L(r) r^{p-1}\, dr < \infty.$$

Remark 6. Carleson ([3], p. 14) also assumes that H is convex, which induces a strong form of the maximum principle ([3], p. 15) for L-potentials. Since the potentials of our kernel L satisfy a weak form of the maximum principle as discovered by Ugaheri ([8], p. 38, Fundamental Lemma), the convexity property of H is not required in the proofs of those results in Carleson's book which we shall need.

Lemma 1. *Let $E \subset \mathbf{R}^p$ $(p \geq 2)$ be a bounded set. If*

$$A = \int_0^\infty L(r)\, dh(r) < \infty. \tag{3.3}$$

there exists a constant depending only on A and p such that

$$M_h(E) \leq \text{const.}\, C_L(E) \tag{3.4}$$

where C_L is the outer capacity function with respect to L.

Proof. We first claim that (3.4) holds for compact sets. If the compact set F is given, let μ be a non-negative set function associated with F such that (3.1) and (3.2) are true. Let

$$u_\mu(x) = \int L(|x-y|)\, d\mu(y) \quad \text{be the } L\text{-potential of } \mu.$$

Arguing as in the second part of the proof of Theorem 1 in Carleson ([3], p. 28), we see that if x_0 is an arbitrary point and if $\Phi(r)=\mu(\{x:|x-x_0|<r\})$,

$$\mu_\mu(x_0) = \int_0^\infty L(r)\, d\Phi(r) \leq \int_0^\infty L(r)\, dh(r) = A.$$

Hence $\mu(F) \leq AC_L(F)$. It follows from (3.2) that

$$M_h(F) \leq \text{const.}\, C_L(F),$$

and we have completed the first step in the proof of Lemma 1.

Secondly, we prove that (3.4) is true for open sets. We use the auxiliary function m_h' defined in Carleson ([3], pp. 6 and 11). It has the following properties: there exist constants C_1 and C_2 depending only on the dimension such that

$$C_1 M_h(E) \leq m_h'(E) \leq C_2 M_h(E).$$

If 0 is open,

$$m_h'(0) = \sup_{F\subset 0} m_h'(F), \ F \text{ compact.}$$

(see [3], §2, (1.3) and (3.6)). Hence, since the lemma is true for compact sets,

$$M_h(0) \leq \text{const. } m_h'(0) \leq \text{const. } \sup_{F\subset 0} m_h'(F) \leq \text{const. } \sup_{F\subset 0} M_h(F)$$
$$\leq \text{const. } \sup_{F\subset 0} C_L(F) = \text{const. } C_L(0).$$

Thus (3.4) holds for open sets.

The result for general bounded sets is now immediate since

$$M_h(E) = \inf_{0\supset E} M_h(0), \qquad 0 \text{ open,}$$

(see [3], p. 9) and

$$C_L(E) = \inf_{0\supset E} C_L(0), \quad 0 \text{ open.}$$

4. **On sets which are minimally thin at** ∞. Let $E\subset K$ be minimally thin at ∞. How can we characterize E as a generalized Azarin set?

THEOREM 5. *If* $E\subset K\subset \mathbf{R}^p$ $(p\geq 2)$ *is minimally thin at* ∞, *then* E *is a generalized Azarin set with respect to any function* h *such that* $\int_0^\infty \phi_p^+(r)\, dh(r) < \infty$.

Proof. Let c be the outer ordinary capacity function in $\mathbf{R}^p$ $(p\geq 2)$. If $p=2$, $c=\gamma\circ\lambda$. Since $c(rE)=r^{p-2}c(E)$ if $p\geq 3$, and $\lambda(rE)=r\lambda(E)$, $p=2$, E is minimally thin at ∞ in $\mathbf{R}^p$ $(p\geq 2)$ if and only if

$$\sum_n c(E_n/s^n) < \infty,$$

(see (1.2) when $p\geq 3$ and (1.3) when $p=2$). By Lemma 1, $M_h(E_n/s^n)\leq \text{Const.}\, c(E_n/s^n)$. Therefore, the minimal thinness of $E\subset K$ at ∞ implies that $\sum M_h(E_n/s^n)<\infty$. For each n, there exists a covering of E_n/s^n by a sequence of balls $\{B_{ni}'\}$, each of radius r_{ni}', such that $\sum_i h(r_{ni}')\leq M_h(E_n/s^n)+2^{-n}$. We can now construct a sequence of $\{B_{ni}\}$ of balls, each of radius $r_{ni}=s^n r_{ni}'$ such that $\{B_{ni}\}$ covers E_n and $r_{ni}/R_{ni}\leq r_{ni}'$ for each i. Since h is non-decreasing, $h(r_{ni}/R_{ni})\leq h(r_{ni}')$, and therefore

$$\sum_i h(r_{ni}/R_{ni}) \leq M_h(E_n/s^n)+2^{-n}.$$

Summing over n, we get a double sequence of balls $\{B_{ni}\}$ which covers E such that the corresponding generalized Azarin sum is finite. The proof of Theorem 5 is complete.

Corollary 1. *Let $\alpha>p-2$ be given. If $E\subset K$ is minimally thin at ∞, then E is a generalized Azarin set of order α.*

Proof. Choosing

$$h(r)=\begin{cases} r^{\alpha}, & 0\leq r\leq 1,\\ 1, & r>1,\end{cases}$$

it is easily checked that if $\alpha>p-2$, $\int_0^\infty \phi_p^+(r)\,dh(r)<\infty$, and the result is immediate from Theorem 5.

We introduce

$$\gamma_\alpha(r)=\begin{cases}\{\log(1/r)\}^{-\alpha}, & 0<r<e^{-1},\\ 1 \quad , & r\geq e^{-1}.\end{cases}$$

Corollary 2. *Let $\alpha>1$ be given. If $E\subset K\subset \mathbf{R}^2$ is minimally thin at ∞, then E is a generalized Azarin set with respect to γ_α.*

Proof. Choosing $h=\gamma_\alpha$, it is easily checked that if $\alpha>1$, $\int_0^\infty \phi_2^+(r)\,d\gamma_\alpha(r)<\infty$, and the result follows directly from Theorem 5.

Remark 7. It is clear from these corollaries that a subset E of a Stolz domain which is minimally thin at ∞, is much smaller than most of those sets which can be covered by the coverings considered by Azarin ([1]).

5. **On radial projections.** Let $E\subset D$ be minimally thin at ∞. According to a result of Mme. Lelong (see [6], p. 132), the radial projection onto the unit sphere of those rays through the origin whose intersection with E have ∞ as a limit point has ordinary capacity zero. What can be said about the radial projection onto the unit sphere of those rays through the origin which hit infinitely many of the balls in a generalized Azarin covering of E with respect to a given function h?

We need one more concept. Let $h=[0,\infty)\rightarrow[0,\infty)$ be a continuous, non-decreasing function such that $h(0)=0$. If $\rho>o$ is given, let

$$\Lambda_h^{(\rho)}(E)=\inf\sum h(r_n)$$

where the set E is covered by a set of balls of radii $\{r_n\}$ where $r_n\leq\rho$ for all n. The limit $\Lambda_h(E)=\lim_{\rho\rightarrow 0}\Lambda_h^{(\rho)}(E)$ is the classical Hausdorff measure of E (see Carleson [3], p. 6).

Theorem 6. (i) *Let $E\subset D\subset\mathbf{R}^p$ $(p\geq 2)$ be the union of a sequence of balls $\{B_n\}$ which has ∞ as a limit point and such that $\sum h(r_n/R_n)<+\infty$. If E^* is the projection onto the unit sphere of those rays whose intersections with E have ∞ as a limit point then $\Lambda_h(E^*)=0$.*

(ii) *Let $g\colon[0,+\infty)\rightarrow[0,+\infty)$ fulfil the same conditions as h and satisfy the inequality $h\leq g$. If there exists a subset $G\subset D\cap\{P:|P|=1\}$ such that $\Lambda_g(G)>0$ and $\Lambda_h(G)=0$, then there exists a union E of a sequence of balls $\{B_n\}$ as described in* (i) *such that $G\subset E^*$ and such that*

$$\sum h(r_n/R_n)<+\infty. \tag{5.1}$$

$$\sum g(r_n/R_n)=+\infty. \tag{5.2}$$

Proof. The radial projections of the balls $\{B_n\}$ onto the unit sphere have radii $\{r_n/R_n\}$. If $\rho>0$ is given, E^* can be covered by balls whose radii $\{r_n/R_n\}_q^\infty$ are all $\leq\rho$, and such that

$$\Lambda_h^{(\rho)}(E^*) \leq \sum_q^\infty h(r_n/R_n).$$

Since the right hand member can be made arbitrarily small by choosing q arbitrarily large, it follows that $\Lambda_h(E^*)=0$, and hence (i) is proved.

If B is a ball of radius r and centre P, let T_nB be a ball of radius s^nr and of centre s^nP. Since $\Lambda_h(G)=0$, if n is given, we can cover G by a v-sequence of balls $\{B'_{nv}\}$ whose radii $\{r'_{nv}\}$ are all less than 2^{-n} and such that

$$\sum_v h(r'_{nv}) < 2^{-n}.$$

We define $B_{nv}=T_nB'_{nv}$ and $E=\bigcup_{n,v} B_{nv}$. It is clear from the construction that the sum (5.1) associated with this set converges. Since $G\subset E^*$ and $\Lambda_g(G)>0$, it follows from (i) that the sum (5.2) associated with the set E diverges. We have proved (ii) and hence our theorem.

As a corollary, we can characterize the size of projections of the type discussed here in terms of capacity. If $\alpha>0$ is given, let C_α be the capacity with respect to the kernel $|P|^{-\alpha}$.

Corollary 1. *If E is a generalized Azarin set of order $\alpha<p-1$ then $C_\alpha(E^*)=0$.*

Proof. It follows from the first part of Theorem 6 that $\Lambda_h(E^*)=0$ where $h(r)=r^\alpha$, $\alpha<p-1$. By making use of the first part of Theorem 1 in Carleson ([3], p. 28) it follows immediately that $C_\alpha(E^*)=0$.

Remark 8. If $p-1\leq\alpha<p$ then Corollary 1 is still valid but it is no longer significant because $C_\alpha(S)=0$ in this case where S is the unit sphere.

Corollary 2. *For each $\alpha\leq p-1$, there exists a generalized Azarin set E of order α, such that $C_\beta(E^*)>0$ for all $\beta<\alpha$.*

Proof. Let $h(r)=r^\alpha$ and choose $g\geq h$ such that

$$r^{-\beta}g(r)\to 0, \qquad r\to 0, \quad \text{for all } \beta, \quad 0<\beta<\alpha. \tag{5.3}$$

$$r^{-\alpha}g(r)\to\infty, \qquad r\to 0, \tag{5.4}$$

$$\text{If} \quad L(r)=(g(r))^{-1}, \qquad \bar{L}(r)\leq \text{Const.}\, L(r), \tag{5.5}$$

for all r sufficiently small.

For the definition of $\bar{L}$, see ([3], Theorem 1, p. 28). One possibility is to take $g(r)=r^\alpha(1+\psi(r)^2)$, where $\psi(r)=\max(\log 1/r, 1)$.

From (5.4) and (5.5), we see that the conditions for Theorem 4 in Carleson ([3], p. 34) are satisfied. It follows that there exists a subset G of the unit sphere such that $C_L(G)>0$ and $\Lambda_h(G)=0$. It is clear from Carleson ([3], Theorem 1,

p. 28), that $\Lambda_g(G)=\infty$. It is also clear from (5.3) and (5.5) that $C_\beta(G)>0$, all $\beta<\alpha$. We can now apply the second part of Theorem 6 to construct the set E with the desired properties.

COROLLARY 3. *For each h such that $\int_0^\delta (h(r)/r)\,dr<\infty$, there exists a generalized Azarin set E with respect to h such that $C_\psi(E^*)>0$, where $\psi(r)=\max(\log 1/r, 1)$.*

Proof. Apply Theorem 5 of Carleson ([3], p. 35) and then repeat the argument of Corollary 2.

REMARK 9. It is clear from Corollary 3 of Theorem 6 along with Mme. Lelong's radial limit theorem for minimally thin sets at ∞ in $\mathbf{R}^2$ (see [6], p. 132) that the implication proved in Theorem 5 is strict in general when $p=2$. One cannot hope therefore to characterize minimally thin sets at ∞ as an Azarin set with respect to some function h such that $\int_0 \phi_p^+(r)\,dh(r)<\infty$.

6. **On classically thin sets in $\mathbf{R}^2$.** A set $E\subset\mathbf{R}^2$ is thin at 0 in the *classical* sense, or simply thin at 0, if and only if there exists a positive superharmonic function u in some neighbourhood of 0 such that $u(0)<+\infty$, whereas $\lim_{\substack{x\to 0\\ x\to E}} u(x)=+\infty$. If I_n, $c_n=c(E_n)$, $\lambda_n=\lambda(E_n)$ are defined as in section 1 but subject to the modification that $0<s<1$, then E is thin at 0 if and only if $\sum_n nc_n<+\infty$ (see [2], p. 81) or $\sum_n \gamma(\lambda_n{}^{1/n})<+\infty$, since $c_n=\gamma(\lambda_n)$ and $\gamma\{(x)^{1/n}\}=n\gamma(x)$. Some elementary calculations show that E is thin at 0 if and only if

$$\sum_n n\gamma\left\{\frac{\lambda_n}{s^n}\right\}<+\infty. \tag{6.1}$$

We recall (see [5], Theorem 1′) that E is minimally thin at 0 with respect to a half plane if and only if $\sum_n \gamma\{\lambda_n/s^n\}<+\infty$, provided that E is restricted to a Stolz domain. It is now clear that thin sets at 0 which are restricted to a Stolz domain are necessarily minimally thin at 0 with respect to a half plane and that this implication is strict in general. Since ordinary thinness is invariant with respect to the inversion mapping, E is thin at ∞ in $\mathbf{R}^2$ if and only if

$$\sum_n n\gamma\circ\lambda\left\{\frac{E_n}{s^n}\right\}<+\infty,\quad\text{or}\quad \sum_n n\gamma\left\{\frac{\lambda_n}{s^n}\right\}<\infty, \tag{6.2}$$

where it is again required that $s>1$ as in earlier sections. We shall now prove some covering theorems for thin sets at ∞ in $\mathbf{R}^2$ which are analogous to our earlier theorems for minimally thin sets at ∞ with respect to the half plane.

THEOREM 7. (*Analogue of Theorem* 2). *Let $E=\bigcup_{n=1}^\infty B_n$ be defined as in Theorem* 2. *Then E is thin at ∞ if and only if*

$$\sum_n n\gamma\left\{\frac{r_n}{R_n}\right\}<+\infty. \tag{6.3}$$

Proof. Since $s^n \leq R_n < s^{n+1}$ we can repeat the reasoning in the proof of Theorem 2 to show that the series $\sum_n n\gamma\{\lambda_n/s^n\}$ and $\sum_n n\gamma\{\lambda_n/s^{n+1}\}$ are co-convergent which in turn implies that E is thin at ∞ if and only if $\sum_n n\gamma\{r_n/R_n\} < +\infty$.

THEOREM 8. (*Analogue of Theorem* 4). *Assume that* $E \subset \mathbf{R}^2$ *can be covered by a sequence of balls* $\{B_n\}$ *where each* $B_n = \{z : |z - z_n| \leq r_n,\ |z_n| = R_n\}$ *and* $R_n \to \infty$ *as* $n \to \infty$. *If*

$$\sum_n (\log R_n)\gamma\left\{\frac{r_n}{R_n}\right\} < +\infty, \tag{6.4}$$

then E is thin at ∞ *in* $\mathbf{R}^2$.

Proof. We define the double sequence $\{B_{nj}\}$ as in the proof of Theorem 4. Then the argument in the proof of Theorem 4 may be repeated to obtain $\sum_{n=2}^{\infty} n(\log s)\gamma\{\lambda_n/s^n\} \leq 3 \sum_{n,j} (\log R_{nj})\gamma \circ \lambda\{E_{nj}/s^n\} < +\infty$. The theorem follows.

THEOREM 9. (*Analogue of Theorem* 5). *If E is thin at* ∞ *in* $\mathbf{R}^2$ *and if h is defined so that* $\int_0 \phi_p^+(r)\,dh(r) < +\infty$, *then there exists a sequence of disks* $\{B_n\}$ *which covers E such that* $\sum_n (\log R_n)h\{r_n/R_n\} < +\infty$.

Proof. Now E is thin at ∞ in $\mathbf{R}^2$ if and only if $\sum_n nc\{E_n/s^n\} < +\infty$, and if we apply Lemma 1 of section 3 it follows that $\sum_n nM_h\{E_n/s^n\} < +\infty$. We now proceed as in the proof of Theorem 5 except that we now define the covering sequence $\{B'_{ni}\}$ of E_n/s_n such that $1 \leq R'_{ni} \leq s$. The covering sequence $\{B_{ni}\}$ for each E_n will necessarily satisfy the restrictions that $s^n < R_{ni} < s^{n+1}$. It follows that $\sum_i h\{r_{ni}/R_{ni}\} < M_h\{E_n/s^n\} + 2^{-n}$ and therefore

$$\textstyle\sum_i (\log R_{ni})h\,\{r_{ni}/R_{ni}\} < (n+1)\log s[M_h\,\{E_n/s^n\} + 2^{-n}].$$

If we now sum over n it is clear that the double sequence $\{B_{ni}\}$ satisfies the requirements of the theorem.

REMARK 10. We can conclude from Theorem 7 that a generalized Azarin set with respect to γ is not necessarily thin at ∞ in $\mathbf{R}^2$. We pointed out in Remark 5 that the implication proved in Theorem 3 is strict in general. We shall now demonstrate by example that the implication proved in Theorem 4 is also strict and that ordinary thin sets at ∞ in $\mathbf{R}^2$ are non-comparable in general with generalized Azarin sets with respect to γ. In defining γ we let $s = e$ and note that $1/\gamma(r) = \psi(r) = \max\{\log(1/r), 1\}$. We shall now construct a one-dimensional Cantor type of set $F \subset [0, 1]$ such that $C_\psi(F) = 0$ whereas $\Lambda_\gamma(F) > 0$ and note that the basic general idea is mentioned in Carleson ([3], p. 35).

We now proceed as in ([3], p. 31). Let $F = \bigcap_{n=1}^{\infty} F_n$ be the usual type of Cantor set with the special requirement that each F_n consists of 2^n closed intervals each of length $\varepsilon_n = 2^{-2^n}$. We write $F_n = \bigcup_{j=1}^{2^n} (J_{nj})$ where each J_{nj} is a component interval of length ε_n.

Lemma 2. *The Cantor set F that we have constructed above possesses the property that* $C_\psi(F)=0$ *whereas* $\Lambda_\gamma(F)>0$.

Proof. Since $\sum_n 2^{-n}\psi(\varepsilon_n)=\sum_n (2^{-n})2^n(\log 2)=\infty$ it follows from Carleson ([3], p. 31, Theorem 3) that $C_\psi(F)=0$. Let J_n be one of the component intervals of F_n. Then if $m>n$, $\sum \gamma(\varepsilon_m)=\gamma(\varepsilon_n)$, where we sum over those intervals of F_m which are contained in J_n. (In the case $m=n+1$, it is obvious that $2\gamma(\varepsilon_{n+1})=\gamma(\varepsilon_n)$. The general case follows by an induction argument). We claim that $m_\gamma(F)>0$, where m_γ is defined as in Carleson ([3], p. 6–7). In order to see this we observe that if F is covered by a finite collection of intervals of length $\{r_v\}_1^q$ such that $\min\{r_v\}\geq 2^{-2^n}$ and such that $r_v=2^{-p_v}$, $p_v=1, 2, 3, \ldots$, then $\sum_v \gamma(r_v)\geq 2^n\gamma(\varepsilon_n)=(\log 2)^{-1}$. Because of the equivalence of m_γ and M_γ (see [3], p. 7), we can therefore conclude that $M_\gamma(F)>0$, and hence $\Lambda_\gamma(F)>0$ which proves our lemma.

Remark 11. If we constructed a Cantor set F such that each $\varepsilon_n=2^{-2^n/n}$, then a slight modification of the proof of Lemma 2 shows that $C_\psi(F)=0$ but $\Lambda_\gamma(F)=\infty$.

Theorem 10. *We can construct a set* $E\subset \mathbf{R}^2$ *which is thin at* ∞ *but is not a generalized Azarin set with respect to* γ.

Proof. Let E' be the image of the Cantor set F as defined in Lemma 2 under a standard one-one mapping which carries the unit interval $[0, 1)$ onto the unit circle. We can still say that $C_\psi(E')=0$ and $\Lambda_\gamma(E')>0$. We now let $E_n=T_n(E')$ where we recall that T_n is defined in the proof of Theorem 6. Then $E=\bigcup_{n=1}^\infty E_n$ has the property that $E^*=E'=E_n^*$ for all n. Hence $c(E_n^*)=0$, and therefore $\sum_n nc(E_n^*)<+\infty$, which implies that E is thin at ∞. We also have $\Lambda_\gamma(E^*)>0$, and hence can conclude from Theorem 6 that E is not a generalized Azarin set with respect to γ.

Remark 12. We can conclude from Theorems 7 and 10 that ordinary thin sets at ∞ in $\mathbf{R}^2$ are non comparable with generalized Azarin sets with respect to γ. This observation trivially implies that the implication of Theorem 4 is also strict.

Remark 13. There is no difference between a classically thin set E at 0 in $\mathbf{R}^p$ $(p\geq 3)$ and a minimally thin set at 0 with respect to a half space, if E is restricted to a Stolz domain [see [2], p. 150). The situation is somewhat different at ∞ because minimally thin sets are preserved by inversion maps whereas classically thin sets are not (see [2], p. 148). At ∞ in $\mathbf{R}^p$ $(p\geq 3)$, classically thin sets are necessarily minimally thin but the converse does not hold in general. A necessary and sufficient condition that E be thin at ∞ in $\mathbf{R}^p$ $(p\geq 3)$ is that $\sum_n c(E_n)<\infty$. For such sets an analogue of Theorem 1 would be that $\sum_n (r_n)^{p-2}<\infty$. Theorems 3 and 9 have similar analogues.

Some of these results have been announced in the Compt. Rend. Acad. Sci. Paris, t. 277 (30 juillet 1973), Série A, 241–242.

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A NOTE ON UPPER BOUNDS FOR THE EIGENVALUES OF $y''+\lambda py=0$

BY
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The natural modes of a small planar transversal vibration of a fixed string of unit length and tension are determined by the eigenvalues and associated eigenfunctions of the differential equation

$$y''(x)+\lambda p(x)y(x) = 0 \tag{1}$$

subject to the boundary condition

$$y(0) = y(1) = 0 \tag{2}$$

where the non-negative function p describes the mass distribution of the string. That the distribution of mass on the string influences the modes of vibration, may be reflected by observing that the eigenvalues determined by the system (1-2) may be considered functions of the density p, $\lambda_n(p)$, where $\lambda_1(p)<\lambda_2(p)<\dots$. It is thus a natural problem to investigate the restrictions imposed upon the eigenvalues $\lambda_n(p)$ when the density p is restricted to a specific class of measurable functions. The main result which we derive is the establishment of an upper bound for the nth eigenvalue $\lambda_n(p)$ when p is a class of functions $E(M, h, H)$ which we define in the following. Our result is a generalization of results of Krein [11, Theorem 4] and Schwarz [12, Theorem 3]. Krein investigated the extremum of the eigenvalues $\lambda_n(p)$ when p is restricted to the class of bounded measurable functions having total mass equal to a fixed constant, M, and Schwarz considered the restriction to piecewise continuous equimeasurable densities. Others [1, 2, 3, 4, 5] have examined bounds for the eigenvalue $\lambda_n(p)$ when the function p is restricted to proper subclasses of the class of functions considered by Krein and when the density p is not necessarily bounded but satisfies restrictions on its form such as being convex, increasing, or $\int_0^x p(t)\,dt$ being concave.

We consider a class of measurable functions defined on [0, 1] which is a natural extension of the class considered by Krein, $E(M, h, H)$, defined for each choice of measurable functions h and H and constant M satisfying

$$0 \le h(x) \le H(x) \quad \text{for} \quad x \in [0, 1],$$

and

$$\int_0^1 h(x)\,dx < M < \int_0^1 H(x)\,dx,$$

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by

$$E(M, h, H) = \left\{p \mid h(x) \le p(x) \le H(x), \int_0^1 p(x)\,dx = M\right\}.$$

For $p \in E(M, h, H)$ $\lambda_n(p)$ satisfies the inequalities $\lambda_n(H) \le \lambda_n(p) \le \lambda_n(h)$, when $h \not\equiv 0$ as the eigenvalues of (1) are known to vary inversely as the density function varies pointwise [8]. Thus $\lambda_n(p)$ is bounded below for any choice of H and h. However, as h approaches the zero function the nth eigenvalue $\lambda_n(h)$ approaches $+\infty$. Thus the least upper bound for $\lambda_n(p)$ as p varies over $E(M, h, H)$ conceivably could increase to infinity if the function h were decreased to the zero function. That this is not the case, and that in fact there exists an upper bound for $\lambda_n(p)$ entirely dependent on the function H and independent of h is given by the

THEOREM. *Let $\lambda_n(p)$ denote the nth eigenvalue of a vibrating string under unit tension, with fixed end points, and mass density given by the non-negative measurable function p. If p is in the class of functions defined on* [0, 1] *having total mass M and bounded pointwise above by a measurable function H and below by a non-negative measurable function h, then $\lambda_n(p) \le \lambda_n^*$ where λ_n^* is a function of H and n only.*

Before proving the Theorem we observe that if the upper boundary function H is bounded with $\mathscr{H} = \sup H(x) < \infty$, $x \in [0, 1]$, then our result is contained in those of Krein [11, Th. 4] and Schwarz [12, Th. 3]. Explicitly Krein showed that in this case

$$\lambda_n(p) \le H(\pi n/M)^2.$$

Thus the principle significance of our result lies in its application in the situation where the function H is not bounded. In this case it seems appropriate to comment on the existence of eigenvalues associated with the functions in $E(M, h, H)$, the function H, and other functions derived from H in the following. Generally a solution of equation (1) is assumed to be a piecewise continuously differentiable function y which satisfies the equation (1) throughout the interval [0, 1]. However, since a measurable function p is determined only up to a set of measure zero, it does not make sense to ask that (1) hold for all $x \in [0, 1]$. Thus instead of requiring a solution to satisfy (1) we ask that $\theta = yp^{1/2}$ satisfy the equivalent integral equation

$$\theta(x) = \lambda \int_0^1 G(x, \xi)(p(x))^{1/2}(p(\xi))^{1/2}\theta(\xi)\,d\xi \tag{3}$$

where G is the Greens function of the system $-y''(x) = 0$, $y(0) = y(1) = 0$. In this sense, since the equation (3) is of Fredholm type with a symmetric L^2 kernel, the system (1-2) will have a sequence of eigenvalues and associated eigenfunctions possessing the same properties as those obtained in the usual sense when p is assumed continuous. (See Tricomi [13, section 3.13].) In particular, as noted by Krein [11, (1.1)], for any non-negative measurable density p defined on $[0, l]$, the first eigenvalue of (1) on $[0, l]$ subject to zero boundary conditions is given by

$$\lambda_1(p) = \min_{\psi\in C^1}\left(\int_0^l [\psi'(x)]^2\,dx \Big/ \int_0^l \psi^2(x)p(x)\,dx\right) \tag{4}$$

where C^1 is the class of continuously differentiable functions ψ defined on $[0, l]$ and satisfying $\psi(0)=\psi(l)=0$.

Proof. Let ρ be a function in $E(M, h, H)$ and denote by u_n the nth eigenfunction of the system (1-2) with $p=\rho$. The function u_n partitions the interval $[0, 1]$ into n nodes determined by its zeros $0=x_0<x_1<\cdots<x_n=1$. There exists at least one choice of the index i such that the interval $I=(x_i, x_{i+i})$ satisfies the inequality $\int_I \rho(x)\,dx\geq M/n$. We then denote by J, the largest subinterval of I, (x_i, x_i+S) such that $\int_J \rho(x)\,dx=M/n$. The nth eigenvalue $\lambda_n(\rho)$ of the system (1-2) with $p=\rho$ is then the first eigenvalue of the equation (1) with the boundary conditions $y(0)=y(x_{i+1}-x_i)=0$ and $p(x)=\rho(x_i+x)$. By further constraining the problem we cannot decrease the eigenvalues and thus we have

$$\lambda_n(\rho) \leq \mu(\rho_0)$$

where $\mu(p)$ denotes the first eigenvalue of the equation (1) subject to the conditions $y(0)=y(S)=0$, and $\rho_0(x)\equiv\rho(x_i+x)$, Courant [8, p. 408].

Denote by ρ_0^+ the symmetrically increasing rearrangement of ρ_0 on the interval $[O, S]$. (See Beesack and Schwarz [7] and [10].) We next invoke a Theorem due to Beesack and Schwarz [7, Th. 2] to obtain the inequality

$$\mu(\rho_0) \leq \mu(\rho_0^+).$$

We note that Beesack and Schwarz stated their Theorem 2 for continuous densities, but utilizing the characterization (4) and noting that the Theorem 378 of [10] which they employed in their proof is applicable to measurable functions, their proof extends to measurable densities as considered here.

We next let H^+ denote the symmetrically increasing rearrangement of the function H over the interval $[0, 1]$. Let t_n denote the least value of t such that $\int_0^t H^+(x)\,dx=M/2n$. We then have

$$\frac{M}{n} = 2\int_0^{t_n} H^+(x)\,dx \geq \int_{x_i}^{x_i+2t_n} H(x)\,dx \geq \int_{x_i}^{x_i+2t_n} \rho(x)\,dx.$$

Consequently, if we had $2t_n>S$, it would then follow from the above that

$$\frac{M}{n} > \int_{x_i}^{x_i+S} \rho(x)\,dx = \frac{M}{n}.$$

Hence $2t_n\leq S$ and we can define the symmetric function H_0 over $[O, S]$ by

$$H_0(x) = \begin{cases} H^+(x), & \text{if } 0\leq x\leq t_n, \\ 0, & \text{if } t_n < x \leq S/2, \\ H_0(S-x), & \text{if } S/2 < x \leq S. \end{cases}$$

Since $\rho(x) \leq H(x)$ for $x \in J$ we have

$$\rho_0^+(x) \leq H_0(x) \quad \text{for} \quad x \in [O, t_n] \cup [S-t_n, S]$$

and clearly $\rho_0^+(x) \geq H_0(x)$ for $x \in (t_n, S-t_n)$. Consequently we may apply Theorem III of Beesack [6] to the functions H_0 and ρ_0^+ on the interval $[O, S]$ to obtain the inequality

$$\mu(\rho_0^+) \leq \mu(H_0).$$

Again, although Beesack stated his Theorem III for continuous densities, his proof extends to cover the case of measurable densities as considered here by utilizing the characterization (4) and noting that Beesacks Theorem II, utilized in his proof of Theorem III, applies to measurable functions.

Finally, we denote by λ_n^* the first eigenvalue of the system

$$\begin{aligned} u''(x)+\lambda_n^* H^+(x)u(x) &= 0, \qquad x \in [0, t_n] \\ u(0) = u'(t_n) &= 0. \end{aligned} \tag{5}$$

Then, due to the symmetry of H_0, we have

$$\mu(H_0) = \lambda_n^*.$$

Combining the above we obtain our result

$$\lambda_n(\rho) \leq \lambda_n^*.$$

The number λ_n^* is not dependent on the choice of ρ and since t_n is a monotone decreasing function of n, we see that to increase n tends to further constrain the system (5) and thus λ_n^* is monotone increasing in n.

The bound λ_n^* given in the above theorem will not generally provide a sharp upper bound for $\lambda_n(p)$ as p varies over $E(M, h, H)$ unless $n=1$ and H is symmetrically decreasing. However the apriori existence of such a bound, valid for any choice of the function h, allows the application of Theorems 3 and 4 of Gentry and Banks [9]. These theorems then assert that for any differentiable function $f(x_1, x_2, \dots, x_n)$, the functional $F(p)=f(\lambda_1(p), \lambda_2(p), \dots, \lambda_n(p))$ will actually assume its supremum on the class $E(M, h, H)$ at a density ρ which is itself extremal in the sense that $\rho(x)$ is equal to either $H(x)$ or $h(x)$ for each $x \in [0, 1]$. This generalizes the results of Krein [11, Th. 5].

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PERMANENTS OF (0, 1)-MATRICES HAVING AT MOST TWO ZEROS PER LINE

BY

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SUMMARY. Let U_n denote the nth ménage number. Within the class of order n matrices of zeros and ones with at most two zeros in every row and column the minimum permanent is U_n when n is even and $-1+U_n$ when n is odd.

If $A=(a_{ij})$ is a real order n matrix, the permanent of A (per A) is defined to be $\sum a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)}$, the sum being over all permutations $\pi\in S_n$, the symmetric group on n letters. Permanents have considerable combinatorial interest, a result in part due to the fact that

$$\text{(1)}\qquad \operatorname{per} A = \operatorname{per} B$$

when A, B are "combinatorially equivalent", i.e., when there exist permutation matrices P, Q such that $B=PAQ$. For example, the "problème des ménages" asks for the number (M_n) of ways $2n$ symbols $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ may be arranged in a circle so that the a's and b's occur alternately but a_i, b_i are not adjacent for any $i=1, 2, \ldots, n$. The answer is $M_n=2n!\,U_n$ where an explicit formula for the nth ménage number U_n is known [4] but equivalently, it may be defined as the permanent of the order n (0, 1)-matrix having exactly two zeros in every line (row or column),

$$\text{(2)}\qquad \begin{bmatrix}
0 & 0 & & & & & & & & & \\
0 & 1 & 0 & & & & & & & 1 & \\
 & 0 & 1 & \cdot & & & & & & & \\
 & & \cdot & \cdot & \cdot & & & & & & \\
 & & & \cdot & \cdot & \cdot & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & & \cdot & \cdot & \cdot & & & \\
 & & & & & & \cdot & \cdot & \cdot & & \\
 & & & & & & & \cdot & 1 & 0 & \\
 & 1 & & & & & & & 0 & 1 & 0 \\
 & & & & & & & & & 0 & 0
\end{bmatrix}$$

or of any combinatorially equivalent matrix [3].

Further interest in permanents is fostered by various unresolved conjectures concerning maximum and minimum permanent values within certain classes of

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matrices, e.g., [2]. In particular, the van der Waerden conjecture remains unresolved [1]. Let $\vartheta(n, k)(\mathscr{B}(n, k))$ denote the class of all order n $(0, 1)$-matrices having exactly $n-k$ (at most $n-k$) zeros per line. It is the purpose of this paper to determine the minimum permanent within $\mathscr{B}(n, n-2)$. Specifically we shall show:

THEOREM 1. *The minimum permanent in* $\mathscr{B}(n, n-2)$ *is* U_n *for* n *even and* $-1+U_n$ *for* n *odd.*

Before turning to the proof we shall make a simplification (Lemma 1) and obtain some useful formulae (Lemma 2). Use will be made of the fact that members of $\vartheta(n, n-2)$ are combinatorially equivalent to $(0, 1)$-complements of the direct sum of $(0, 1)$-complements of matrices of type (2) of orders $p_1, p_2, \ldots, p_\nu$ where $n=p_1+p_2+\cdots+p_\nu$ is a partition of n with all $p_i \geq 2$.

LEMMA 1. *The minimum permanent in* $\mathscr{B}(n, n-2)$ *can be found in the union of the following two subclasses of* $\mathscr{B}(n, n-2)$:

(i) $\vartheta(n, n-2)$

(ii) *the class of order* n *matrices of the form:*

$$\left[\begin{array}{c|ccccc} 0 & 1 & 1 & \cdot & \cdot & \cdot\ 1 \\ \hline 1 & & & & & \\ 1 & & & & & \\ \cdot & & & B & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & & & & & \end{array}\right] \tag{3}$$

with $B \in \vartheta(n-1, n-3)$. (*Note, since the constant line sum of* B *is two less than its order, all combinatorially inequivalent forms for* B *have been implicitly described above.*)

Proof. If A has minimum permanent within the class $\mathscr{B}(n, n-2)$ then we can assume that every one in A is in some line with sum exactly equal to $n-2$, else such a one can be removed without change in the permanent value. For convenience we shall say such a reduced A has "property R". Now suppose $A \notin \vartheta(n, n-2)$. Clearly A cannot have a line, say a row, of sum n since by property R every column would contain two zeros and hence since the matrix is square, some row would contain more than two zeros, i.e., $A \notin \mathscr{B}(n, n-2)$. It follows that A has both a row and a column with sum $n-1$. Let the first column of A be such a column, taking its zero to be in the first row. By property R the last $n-1$ rows have row sums equal to $n-2$ so the first row must be the one with sum $n-1$. Hence A is of the form (3) with the row sums of B equal to $n-3$. Repeating the argument for the column sums of B we conclude $B \in \vartheta(n-1, n-3)$.

It is obvious that an even simpler application of property R solves the analogous problem of the minimum permanent in $\mathscr{B}(n, n-1)$. Namely, the minimum permanent in this class is D_n, the nth derangement number, for which an explicit formula is known but which can be equivalently defined as the permanent of the (0, 1)-complement of the order n identity matrix [3].

By a "list" we mean a finite unordered set of positive integers. A particular list of length μ will be denoted either as

$$[p_1, p_2, \ldots, p_\mu] \tag{4}$$

or, more commonly as

$$(p_2, p_2, \ldots, p_\mu). \tag{5}$$

The notation (5) will be used rather than (4) when some terms have possibly been suppressed, i.e.,

$$(p_1, p_2, \ldots, p_\mu) = [p_1, p_2, \ldots, p_\mu, p_{\mu+1}, \ldots, p_{\mu+\nu}]$$

for some non-negative integer ν and for some positive integers $p_{\mu+1}, \ldots, p_{\mu+\nu}$. By $U[p_1, p_2, \ldots, p_\mu]$ we shall denote the permanent of the matrix which is the (0, 1)-complement of the direct sum of (0, 1)-complements of matrices of type (2) of orders $p_1, p_2, \ldots, p_\mu$, respectively. If some $p_i = 1$, the corresponding summand will be an order one zero matrix. By (1) the value $U[p_1, p_2, \ldots, p_\mu]$ is independent of the ordering of the p_i's and by Lemma 1 $U[p_1, p_2, \ldots, p_\mu]$, for some partition $n = p_1 + p_2 + \cdots + p_\mu$ of n, is the minimum permanent wanted. $U(p_1, p_2, \ldots, p_\mu)$ has the same definition as $U[p_1, p_2, \ldots, p_\nu]$, the notation implying that indication of some matrix summands might have been suppressed. In particular, an equation involving $U(p_1, p_2, \ldots, p_\mu)$, $U(q_1, q_2, \ldots, q_\nu)$, etc., holds when identical, arbitrary (positive integer) terms are adjoined to all lists.

LEMMA 2. *If* $k > 1$, $l > 1$,

$$U(k, l) = U(k+l) + 2\sum_{i=1}^{k+l-1} U(i) - \sum_{i=1}^{k-1} U(i, l) - \sum_{i=1}^{l-1} U(k, i). \tag{6}$$

If $k = 1$, $l > 1$, (6) *has the modified form*

$$\sum_{i=1}^{l} U(1, i) = 3\sum_{i=1}^{l-1} U(i) + 2U(l) + U(l+1). \tag{7}$$

If $l > 2$,

$$U(1, l) = U(l-1) + U(l) + U(l+1). \tag{8}$$

Proof. If the zero in the (1, 1) position of the matrix A_n in (2) is replaced by a one, the new permanent value, per A_n^1, is

$$U_n + \text{per } A_{n-1}^1$$

so by induction,

$$\text{per } A_n^1 = \sum_{i=1}^{n} U_i. \tag{9}$$

Moreover, this is the permanent of any matrix obtained by replacing a zero of A_n by a one since all such matrices are combinatorially equivalent. Let $k>1$, $l>1$, and consider the matrices which define $U[k, l]$, $U[k+l]$:

$$(10)\qquad \begin{array}{c|cccccc|cccccc|}
 & 0&0&&&&&&&&&& \\
 & 0&1&\cdot&&1&&&&&&& \\
 & &\cdot&\cdot&\cdot&&&&&1&&& \\
k & &&\cdot&\cdot&\cdot&&&&&&& \\
 & &&&\cdot&\cdot&\cdot&&&&&& \\
 & 1&&&\cdot&1&0&&&&&& \\
 & &&&&0&\boxed{0}&\boxed{1}&&&&& \\
 & &&&&&\boxed{1}&\boxed{0}&0&&&& \\
 & &&&&&&0&1&\cdot&&1& \\
 & &&&&&&&\cdot&\cdot&\cdot&& \\
l & &&1&&&&&&\cdot&\cdot&\cdot& \\
 & &&&&&&&&&\cdot&\cdot&\cdot \\
 & &&&&&&1&&&\cdot&1&0 \\
 & &&&&&&&&&&0&0
\end{array}
\qquad
\begin{array}{c|cccccc|cccccc|}
 & 0&0&&&&&&&&&& \\
 & 0&1&\cdot&&1&&&&&&& \\
 & &\cdot&\cdot&\cdot&&&&&1&&& \\
k & &&\cdot&\cdot&\cdot&&&&&&& \\
 & &&&\cdot&\cdot&\cdot&&&&&& \\
 & 1&&&\cdot&1&0&&&&&& \\
 & &&&&0&\boxed{1}&\boxed{0}&&&&& \\
 & &&&&&\boxed{0}&\boxed{1}&0&&&& \\
 & &&&&&&0&1&\cdot&&1& \\
 & &&&&&&&\cdot&\cdot&\cdot&& \\
l & &&1&&&&&&\cdot&\cdot&\cdot& \\
 & &&&&&&&&&\cdot&\cdot&\cdot \\
 & &&&&&&1&&&\cdot&1&0 \\
 & &&&&&&&&&&0&0
\end{array}$$

These matrices differ only in an order 2 submatrix as indicated. By comparing the sets of $k+l$ ones, no two in a line, occurring in the two matrices (10) and by making use of (9), we obtain the identity (6). This fails if $k=1$, but a slight modification of the comparison gives

$$U(1, l)-\sum_{i=1}^{l-1} U(i)-2\sum_{i=1}^{l} U(i) = U(l+1)-\sum_{i=1}^{l-1} U(1, i)$$

which is a rearrangement of (7). If $l-1>1$ we can rewrite (7) with $l-1$ replacing l. Subtraction of these two forms of (7) then gives (8).

By (8) $U(1, l)>U(l+1)$ for $l>2$ so that except possibly for $U(1, 1)$, $U(1, 2)$, permanent values smaller than $U(1, l)$ can be found among the values $U(l+1)$. Theorem 1 will now be proven. First we show $U[p_1, p_2, \ldots, p_\mu]\geq U[q]$ if

$$q = \sum_{i=1}^{\mu} p_i,$$

unless $\mu=2$ and $|p_1-p_2|=1$. Finally we complete the proof by showing $U[k, k+1]=-1+U[2k+1]$.

Proof of Theorem 1. Our first reduction is to show:

$$(11)\qquad U(k, l) = U(l+k)+U(l-k) \quad \text{if} \quad k>1 \quad \text{and} \quad l>k+1.$$

Subtracting expressions for $U(2, 4)$, $U(2, 3)$ as obtained from (6) we have

$$U(2, 4)-U(2, 3) = U(6)+U(5)+U(1, 3)-U(1, 4)-U(2, 3)$$

and using (8) this simplifies to (11) when $k=2$ and $l=4$. The same approach works for $U(k, l)$ by induction. Assume the result for $U(k', l')$ for all pairs $k', l'(k'>1$ and $l'>k'+1)$ satisfying $k'<k$ or $l'<l$ if $k'=k$. Again by (6),

$$U(k, l)-U(k, l-1) = U(k+l)+U(k+l-1)-U(1, l) \tag{12}$$
$$-\sum_{i=2}^{k-1} U(i, l)+U(1, l-1)+\sum_{i=2}^{k-1} U(i, l-1)-U(k, l-1).$$

Using (8) for the terms $U(1, l)$, $U(1, l-1)$ and applying the induction hypothesis to the two summations in (12) we obtain the result wanted.

The proof of Theorem 1 has now been reduced to a consideration of those lists of the form

$$[k, k, \ldots, k, k+1, k+1, \ldots, k+1] \quad \text{with} \quad k \geq 1.$$

We next note that:

$$U(k, k) \geq U(2k) \quad \text{if} \quad k \geq 1. \tag{13}$$

Applying (11) to (6) and making use of (8) (assuming $k \geq 3$), we have

$$U(k, k) = U(2k)+2U(1)+2U(2k-1)-2U(k-1, k) \tag{14}$$

$$U(k, k+1) = U(2k+1)+2U(1)+U(2k-1)+U(2k)-U(k-1, k)-U(k, k). \tag{15}$$

Equating the two expressions for $U(k, k)-U(2k)-2U(1)$ as obtained from (14), (15) we get

$$U(k, k+1)-U(2k+1) = U(k-1, k)-U(2k-1). \tag{16}$$

So (14) becomes

$$U(k, k)-U(2k) = 2\{U(1)+U(5)-U(2, 3)\}, \qquad k \geq 3 \tag{17}$$

after successive applications of (16). For $k=2$ (14), (15) must be modified, but similarly equating the analogous expressions for $U(2, 2)-U(4)-2U(1)$ we obtain

$$U(2, 3)-U(5) = U(1, 2)-U(3)-U(2).$$

Consequently, for $k>1$, (17) becomes

$$U(k, k)-U(2k) = 2\{U(1)+U(2)+U(3)-U(1, 2)\}$$

and using (7) for $l=2$,

$$U(k, k)-U(2k) = 2\{U(1, 1)-2U(1)-U(2)\}.$$

If we consider the terms in the permanent value $U(1, 1)$ we have

$$U(1, 1) = U(2)+2U(1)+U(\ ) \tag{18}$$

and therefore (17), for $k>1$ can be written

$$U(k, k)-U(2k) = 2U(\ )$$

so that $U(k, k)-U(2k) \geq 0$ if $k>1$ and the same result follows from (18) for $k=1$.

To complete the proof of Theorem 1 we note that $U[k, k+1] = -1 + U[2k+1]$ for $k \geq 1$. This is readily checked for $k = 1, 2$ and for $k \geq 3$ (16) gives us

$$U[k, k+1] - U[2k+1] = U[2, 3] - U[5] = -1.$$

A final point should be noted: For all $n \neq 3$, the minimum permanent in $\mathscr{B}(n, n-2)$ is attained in $\vartheta(n, n-2)$.

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QUASI-INJECTIVE AND PSEUDO-INJECTIVE MODULES

BY
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1. Let R be a ring with identity not equal to zero. A right R-module is said to be quasi-injective (pseudo-injective) if for every submodule N of M, every R-homomorphism (R-monomorphism) of N into M can be extended to an R-endomorphism of M [7] ([13]). An example of a pseudo-injective module which is not quasi-injective was given by Hallett ([4]; also see lemma 2 in this paper). It is known (Harada [5]) that a direct sum of finitely many copies of a quasi-injective module is quasi-injective. We show that if a direct sum of two copies of a pseudo-injective module M is pseudo-injective then M is quasi-injective. Hallett [4] and Singh [14] have shown independently that pseudo-injective modules over PID are quasi-injective. In his thesis, Hallett also showed that self-pseudo-injective generalized uniserial rings are self-injective. In this paper we show that any pseudo-injective module over a generalized uniserial ring is quasi-injective (theorem 4) and we use this to show that any torsion pseudo-injective module over a bounded hereditary noetherian prime ring is quasi-injective (theorem 5). An example is given that a pseudo-injective module over an arbitrary hereditary noetherian prime ring need not be quasi-injective. It is also shown that torsion free pseudo-injective modules over prime Goldie rings are injective and this extends an earlier result of Singh [14].

2. LEMMA 1. *A direct summand of a pseudo-injective module is pseudo-injective.*

The proof is obvious.

THEOREM 1. *Let $N_1 \oplus N_2$ be a pseudo-injective module and $\sigma: N_1 \to N_2$ be a monomorphism. Then σ splits and N_1 is quasi-injective.*

Proof. Since $\eta: \sigma(N_1) \to N_1 \oplus N_2$ given by $\eta(\sigma(x)) = (x, 0)$, $x \in N_1$, is a monomorphism, it can be extended to an endomorphism η^* of $N_1 \oplus N_2$. If $q: N_2 \to N_1 \oplus N_2$ and $p: N_1 \oplus N_2 \to N_1$ are natural injection and projection respectively, then $\lambda = p\eta^* q: N_2 \to N_1$ is such that $\lambda\sigma = I_{N_1}$. Hence σ splits. Let $N_1 \oplus N_1' = N_2$. So $N_1 \oplus N_2 = N_1 \oplus N_1 \oplus N_1'$ and $T = N_1 \oplus N_1$ is pseudo-injective by lemma 1. Write $T = M_1 \oplus M_2$, $M_1 = M_2 = N_1$. Let N be any submodule of N_1 and σ: $N \to N_1$ be an R-homomorphism. If we treat N as a submodule of T contained in M_1, then the mapping

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$\eta: N \to T$ given by $\eta(x) = (x, \sigma(x))$, $x \in N$, is a monomorphism. Hence it can be extended to an endomorphism λ of T. If $q_1: M_1 \to T$ and $p_2: T \to M_2$ are natural injection and projection respectively then $\mu = p_2 \lambda q_1$ is an endomorphism of N_1 which extends σ. Hence N_1 is quasi-injective.

COROLLARY. *A module M is quasi-injective if and only if $M \oplus M$ is pseudo-injective.*

The proof follows from theorem 1 and Harada [5].

3. In this section we study pseudo-injective modules over generalized uniserial rings and hereditary noetherian prime rings (hnp-rings). A right and left artinian ring R is said to be generalized uniserial if for every primitive idempotent e of R eR (Re) have unique composition series as right (left) R-modules. These rings have been called serial by Eisenbud and Griffith [1]. A module X of finite composition length is said to be uniserial if it has a unique composition series. A prime ring R which is left noetherian, left hereditary as well as right noetherian, right hereditary is called a hereditary noetherian prime ring (hnp-ring). For results on hnp-rings we refer to Eisenbud-Robson [2], Lenagan [8], and McConnell-Robson [11].

THEOREM 2. (*Nakayama; see also* [1]). *Let R be a generalized uniserial ring. Then every R-module is a direct sum of uniserial modules.*

The above theorem shows that any indecomposable module over a generalized uniserial ring is a uniserial module. Let E and F be two indecomposable modules over a generalized uniserial ring R and let $m(E, F)$ denote the submodule of E which is minimal among the kernels of all homomorphisms of E into F; as E is uniserial, $m(E, F)$ is well defined and unique. $m(E, F) = 0$ if and only if there exists a monomorphism of E into F. For any module X, let $E(X)$ denote its injective hull and $l(X)$ denote the composition length.

THEOREM 3. *Let N be a module over a generalized uniserial ring R. Then N is quasi-injective if and only if $N = \oplus \sum_{i \in \Lambda} N_i$, where N_i are uniserial and $\ell(N_i) \leq l(N_j) + \ell(m(E(N_i), E(N_j))$, $i, j \in \Lambda$.*

Proof. By theorem 2, $N = \oplus \sum N_i$, where N_i are uniserial. Since R is noetherian, by Matlis [10], $E(N) = \oplus \sum E(N_i)$. For convenience let us write E_i for $E(N_i)$. This gives that any R-endomorphism of $E = E(N)$ is determined by R-homomorphisms between various E_i and E_j. Since a module N is quasi-injective if and only if it is invariant under every endomorphism of $E(N)$ (Johnson and Wong [7]), we obtain that $N = \oplus \sum_{i \in \Lambda} N_i$ is quasi-injective if and only if $\sigma(N_i) \subseteq N_j$ for all $\sigma \in \mathrm{Hom}_R(E_i, E_j)$, $i, j \in \Lambda$. Let N be quasi-injective. Let $\sigma: E_i \to E_j$ be an R-homomorphism with $\ker \sigma = m(E_i, E_j)$. If $N_i \subseteq m(E_i, E_j)$ then obviously $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$; otherwise we must have $m(E_i, E_j) \subset N_i$. Then $N_i / m(E_i, E_j) \cong \sigma(N_i) \subseteq N_j$ gives $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$. Conversely, let the inequality hold. Let

$\sigma \in \mathrm{Hom}_R(E_i, E_j)$. Then using the inequality and the fact that $m(E_i, E_j) \subseteq \ker \sigma$, we immediately get $\sigma(N_i) \subseteq N_j$. Hence N is quasi-injective.

An analogous result for quasi-injective modules over hnp-rings which are not right primitive was established by Singh in [16].

We now use the above theorem to obtain one of our main results.

Theorem 4. *A pseudo-injective module over a generalized uniserial ring R is quasi-injective.*

Proof. Let N be a pseudo-injective R-module. By theorem 2, we can write $N = \oplus\sum_{i\in\Lambda} N_i$, where N_i are non-zero uniserial modules. Let $E_i = E(N_i)$. We prove that for all $i, j \in \Lambda$, $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$. Then theorem 3 will yield that N is quasi-injective. Clearly we only need to consider the case when $m(E_i, E_j) \subset N_i$. Now by lemma 1, $N_i \oplus N_j$ is pseudo-injective. Let $\sigma: E_i \to E_j$ be an R-homomorphism with $\ker \sigma = m(E_i, E_j)$. Let F_j be the simple submodule of E_j. Since E_j is uniserial, $F_j \subset N_j$. Also then $\sigma^{-1}(F_j) \subset N_i$. Define $\eta: \sigma^{-1}(F_j) \to N_i \oplus N_j$ by $\eta(x) = (x, \sigma(x))$, $x \in \sigma^{-1}(F_j)$. η is an R-monomorphism; thus it can be extended to an R-endomorphism η^* of $N_i \oplus N_j$. If $\lambda_i: N_i \to N_i \oplus N_j$ and $p_j: N_i \oplus N_j \to N_j$ are natural injections and projections, then $p_j \eta^* \lambda_i: N_i \to N_j$ is such that its restriction to $\sigma^{-1}(F_j)$ is equal to the restriction of σ to $\sigma^{-1}(F_j)$. Thus $\ker(p_j \eta^* \lambda_i) = \ker \sigma = m(E_i, E_j)$. Hence

$$N_i / m(E_i, E_j) \cong (p_j \eta * \lambda_i)(N_i) \subseteq N_j$$

gives that $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$. This proves the theorem.

A ring R is said to be right (left) bounded if each of its essential right (left) ideals contains a non-zero two-sided ideal. A ring R which is both right and left bounded is called bounded.

Theorem 5. *Any torsion pseudo-injective module M over a bounded hnp-ring R, is quasi-injective.*

Proof. Let N be a submodule of M and $\sigma: N \to M$ be an R-homomorphism. We shall show that σ can be extended to an R-endomorphism of M. By an application of Zorn's lemma we suppose that $N \neq M$ and σ cannot be extended to any submodule N' of M containing N properly. Choose $x \in M$ such that $x \notin N$. Now $\mathrm{ann}(x) = \{a \in R \mid xa = 0\}$ is an essential right ideal and so it contains a non-zero two-sided ideal (Eisenbud and Robson [2]). Set $A = \mathrm{ann}(xR)$ which is a non-zero two-sided ideal, and $L = \{y \in M \mid yA = 0\}$. Then L is a module over a generalized uniserial ring R/A. As L is fully invariant submodule of M, L is also pseudo-injective. Hence by theorem 4, L is quasi-injective. Define an R-homomorphism $\lambda: xR \cap N \to L$ by $\lambda(z) = \sigma(z)$, $z \in xR \cap N$. As $xR \subset L$ and L is quasi-injective, λ can be extended to an R-endomorphism λ^* of L. Define $\sigma^*: N + xR \to M$ by $\sigma^*(n + xv) = \sigma(n) + \lambda^*(xv)$. Then σ^* is a well defined R-homomorphism and is a proper extension of σ. This is a contradiction. Hence M is quasi-injective. This completes the proof of the theorem.

REMARK. T. Lenagan [8] has shown that an hnp-ring is either bounded or right as well as left primitive. Thus the above theorem holds for hnp-rings which are not right primitive. The following example due to referee shows that the theorem is not true for an arbitrary hnp-ring.

EXAMPLE. Let Φ be an algebraically closed field and x, y be indeterminates. Let $B=\Phi(y)[x]$ be the hereditary simple principal ideal domain over the field of rational functions $\Phi(y)$ where $xf-fx=df/dy, f\in\Phi(y)$. Then from McConnell and Robson ([11], section 4), we have non-split short exact sequences

$$0\to\frac{B}{(x+y)B}\to\frac{B}{x(x+y)B}\to\frac{B}{xB}\to 0 \tag{1}$$

$$0\to\frac{B}{\left(x+y-\frac{1}{y}\right)B}\to\frac{B}{(x+y)\left(x+y-\frac{1}{y}\right)B}\to\frac{B}{(x+y)B}\to 0 \tag{2}$$

such that

$$\frac{B}{(x+y)B}\approx\frac{B}{\left(x+y-\frac{1}{y}\right)B}(=S,\text{ say})\approx\frac{B}{xB}(=T,\text{ say}). \tag{3}$$

Let $M=B/x(x+y)(x+y-(1/y))B$. Then M is a uniserial module of length 3 having proper submodules

$$M_1=\frac{xB}{x(x+y)\left(x+y-\frac{1}{y}\right)B}\approx\frac{B}{(x+y)\left(x+y-\frac{1}{y}\right)B}$$

and

$$M_2=\frac{x(x+y)B}{x(x+y)\left(x+y-\frac{1}{y}\right)B}\approx\frac{B}{\left(x+y-\frac{1}{y}\right)B};$$

and composition factors T, S, and S. M is pseudo-injective since any monomorphism of M_1 or M_2 is multiplication by an element of Φ (note by theorem 4.1 in [11] that $\mathrm{End}(S)=\mathrm{End}(T)=\Phi$). But M is not quasi-injective since if $\pi:M_1\to(M_1/M_2)\approx M_2$ is a natural homomorphism and $\pi^*\varepsilon\,\mathrm{End}(M)$ is an extension of π, then $(M/\mathrm{Ker}\ \pi^*)=(M/M_2)\approx M_1$ which is not possible as is clear from (1), (2) and (3).

The above example also provides us an example of a uniserial pseudo-injective module which is not quasi-injective.

4. Next let R be a prime Goldie ring with the $n\times n$ matrix ring D_n over a division ring D as its Öre-ring of quotients. Then by Faith-Utumi's theorem ([3], p. 91) there exists a subring K of R such that $K_n\subseteq R\subset D_n$. It was shown by Singh [14] that if M is a torsion free pseudo-injective R-module and $1 \in K$ then M is injective. Here we provide a shorter proof of that theorem and do not assume that $1 \in K$.

THEOREM 6. *Any torsion free pseudo-injective right module M over a prime right Goldie ring R is injective.*

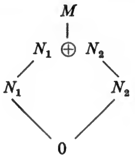
Proof. Firstly we show that M has a non-zero injective submodule. Let Q be the quotient ring of R. It is well known that MQ is the injective hull of R-module M (Levy [9]), and is a direct sum of isomorphic simple Q-submodules N_i. Then $U=N_i \cap M$ is a uniform R-submodule of M. Let V be a uniform submodule of M which is maximal among uniform submodules of M. We proceed to show that V is quasi-injective. Let W be a submodule of V and $\sigma: W \to V$ be a non-zero R-homomorphism. Since W, V are uniform and torsion free, σ is an R-monomorphism. Hence it can be extended to an R-endomorphism η of M. Then η is also mono on V. As both V, $\eta(V)$ are essential extensions of $\eta(W)$ in M, $V+\eta(V)$ is also an essential extension of $\eta(W)$ in M. But V is clearly a maximal essential extension of $\eta(W)$ in M and is unique (Johnson [6]). So $V+\eta(V)=V$. Thus $\eta(V)\subseteq V$ and V is quasi-injective. But then V is indeed injective as we proceed to show. Since V is uniform, its injective hull $E(V)$ is isomorphic to eQ where e is a primitive idempotent of Q. We may regard V as a submodule of eQ. Since $eQe\subseteq \mathrm{Hom}_R(eQ, eQ)$ and V is invariant under every R-endomorphism of eQ [7], we get $eQeV\subset V$. Set $U=V \cap R$. Then $QeU=Q$, since $QeU \cap R$ is a non-zero two-sided ideal of R. Hence $V=eQ$ and thus injective. Now by Zorn's lemma we can find a maximal family (M_i) of injective submodules of M such that ΣM_i is a direct sum. Then M must be equal to the direct sum of this family (M_i). Hence M is injective.

REMARK. Suppose R is an hnp-ring which is not right primitive and M is a pseudo-injective R-module which is not torsion. If the torsion submodule $T(M)$ is a direct summand of M then it can be shown that M is injective.

5. An example of a pseudo-injective module over a simple principal ideal domain which is not a quasi-injective module is given in section 3. In this section we give two additional examples of pseudo-injective modules which are not quasi-injective.

First we prove

LEMMA 2. *Let M be an R-module whose lattice of submodules is*



and N_1 *is not isomorphic to* N_2. *Then the following hold*

(i) M *is not quasi-injective*

(ii) M *is pseudo-injective iff the endomorphism rings of* N_i *are isomorphic to* $Z/(2)$.

Proof. We first note that any non-zero R-endomorphism f of M is either R-automorphism or has its kernel equal to $N_1 \oplus N_2$. From this it follows that the projection of $N_1 \oplus N_2$ to N_1 (or N_2) cannot be extended to an endomorphism of M. Hence M is not quasi-injective.

To prove (ii) let us assume that endomorphism rings $\mathrm{End}(N_i)$ are isomorphic to $Z/(2)$. Since N_i's are non-isomorphic, any monomorphism of N_1, N_2, or $N_1 \oplus N_2$ to N must be an inclusion map and hence can be lifted to the identity map of M. This shows that M is pseudo-injective. Conversely, let M be pseudo-injective. Let f_1, f_2 be two distinct non-zero members of $\mathrm{End}(N_1)$. Define R-homomorphisms $g_i : N_1 \oplus N_2 \to M$ by $g_i(n_1+n_2)=f_i(n_1)+n_2$, $n_1 \in N_1$, $n_2 \in N_2$, $i=1, 2$. Then g_i is an R-monomorphism and can be lifted to an R-endomorphism h_i of M. Set $h=h_1-h_2$. Then $h \neq 0$, h is not an automorphism and $\ker h \neq N_1 \oplus N_2$. This contradicts the assertion made in the beginning of the proof. Hence $\mathrm{End}(N_1) \simeq Z/(2)$. Similarly $\mathrm{End}(N_2) \simeq Z/(2)$. This completes the proof of the lemma.

We now give examples of pseudo-injective modules which are not quasi-injective—One due to Hallett and the other due to Teply. It may be verified that each of these modules has its lattice of submodules isomorphic to the lattice in the lemma 2.

EXAMPLE (Hallett). Let R be an algebra over $Z/(2)$ having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	0	0	n_3	0
e_2	0	e_2	0	n_1	0	0	n_4
e_2	0	0	e_3	0	n_2	0	0
n_1	n_1	0	0	0	0	0	0
n_2	n_2	0	0	0	0	0	0
n_3	0	0	n_3	0	0	0	0
n_4	0	0	n_4	0	0	0	0

Then the right R-module $M=e_2R$ is pseudo-injective but not quasi-injective.

EXAMPLE (Teply). Let $F=Z/(2)$ and $A=F[X]$. Then $A/(x)$ is a $(A/(x)-A/(x^2))$-bimodule in the natural way, and

$$R=\left\{\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \,\middle|\, u, v \in A/(x),\, w \in A/(x^2)\right\}$$

is a ring with usual binary operations. Let M be the right ideal

$$\left\{\begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix} \,\middle|\, v \in A/(x),\, w \in A(x^2)\right\}.$$

Then M_R is pseudo-injective but not quasi-injective.

REMARK. In the above if the field with two elements is replaced by an arbitrary field K then one can verify that the corresponding modules are pseudo-injective if and only if $K \simeq Z/(2)$.

We conclude this paper with some open questions.

Question 1. Let M be a right R-module with right singular submodule $M^{\Delta}=(0)$. Is it true that M is pseudo-injective if and only if M is quasi-injective?

The answer is known to be in the affirmative in the following cases: (i) If $2x=0$, $x \in M$ implies $x=0$ (theorem 3.8, [13]), (ii) If R is a prime right Goldie ring (theorem 6, this paper).

Question 2. Let R be a commutative ring (with identity) and M be a pseudo-injective R-module. Is it true that M is quasi-injective?

We can show that if R is commutative artinian then R_R is pseudo-injective iff R_R is injective. We may assume R is local. Then the minimal ideals are all isomorphic and hence by pseudo-injectivity there is only one minimal ideal. Then by Nakayma's definition of QF-rings [12], R is a QF-ring.

(Added Later) Professor Mark L. Teply in [18] has given a construction for forming pseudo-injective modules which are not quasi-injective. He uses this construction to obtain examples which answer in the negative the above questions 1 and 2.

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CONTRACTION PROPERTY OF THE OPERATOR OF INTEGRATION

BY

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ABSTRACT. It is shown that the operator of integration $Fy(x)=\int_0^x y(t)\,dt$ defined on the space $C(-\infty, \infty)$ of all continuous real valued functions on $(-\infty, \infty)$ is a contraction relative to a certain family of seminorms generating the topology of uniform convergence on compacta. However, as a contrast to this it is proved that F is not contractive with respect to any metric on $C(-\infty, \infty)$ inducing the above topology on $C(-\infty, \infty)$.

1. **Introduction.** Let X be a metrizable topological space and $F:X\to X$ a continuous selfmapping of X into itself. We say F is a *topological contraction* if there is a suitable metric ρ on X inducing the topology of X and a constant $q \in (0, 1)$ such that $\rho(Fx, Fy)\leq q\rho(x, y)$ for all $x, y \in X$.

Assume now X is a Fréchet linear topological space and $F: X\to X$ a linear operator on X satisfying the following condition:

There exists a sequence of seminorms $\{p_n \mid n\geq 1\}$ on X inducing the topology of X and a number $q \in (0, 1)$ such that $p_n(Fx)\leq qp_n(x)$ for all $x \in X$ and all $n=1, 2, \ldots$ It is natural to call such a linear operator F a *generalized contraction* on X. In [1] has been investigated a more general case where X is a completely regular not necessarily metrizable topological space and $F: X\to X$ a contraction with respect to a suitable family of pseudometrics inducing the topology of X.

Since the Fréchet space X is a metrizable topological space a question arises whether a generalized contraction on X is also a topological contraction in the sense of the first definition. The main purpose of this note is to show that the answer is "no", exhibiting at the same time a contraction property of the operator of integration $y(x)\to\int_0^x y(t)\,dt$ in the Fréchet space $C(-\infty, \infty)$. We prove the following.

THEOREM. *Let $C=C(-\infty, \infty)$ denote the linear space of all continuous real valued functions on $(-\infty, \infty)$ endowed with the topology of uniform convergence on compacta, and let $F: C\to C$ be defined by $Fy(x)=\int_0^x y(t)\,dt$ for $y \in C$. Then the operator F is a generalized contraction on C but it is not a topological contraction on C.*

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2. Proof of the theorem.

LEMMA. *Let X be a metrizable topological space and $F:X\to X$ a self-mapping on X such that the following conditions are satisfied:*

(i) *there is a fixed point $x_0\in X$ of F*, i.e., $F(x_0)=x_0$

(ii) *there is a metric ρ on X inducing the topology of X relative to which F is a contraction*, i.e., *there exists a constant $q\in(0,1)$ such that $\rho(Fx,Fy)\le q\rho(x,y)$ for all $x,y\in X$.*

Then there exists an open neighbourhood $U(x_0)$ of x_0 such that for any neighbourhood $V(x_0)$ of x_0 there is an integer $k_0\ge 1$ for which the following implication holds: $k\ge k_0\Rightarrow F^k(U(x_0))\subset V(x_0)$, showing that the iterated images $F^k(U(x_0))$ of $U(x_0)$ under F shrink into any prescribed neighbourhood $V(x_0)$ of x_0 for sufficiently large values of k.

Proof. This is a standard argument.

We are now in the position to prove our theorem. First of all we observe that the topology of C can be induced by the sequence of seminorms defined by

$$\sup_{-n\le x\le n}|f(x)| \quad \text{for any } n=1,2,\dots, \quad \text{and} \quad f\in C.$$

However, the operator F is not contractive with respect to this family. As was done by S. C. Chu and J. B. Diaz in [2] in a different setting, we achieve our end by an elementary modification of the seminorms. Indeed one finds easily that the equivalent family $\{p_n \mid n\ge 1\}$ of seminorms defined by

$$p_n(f)=\sup_{-n\le x\le n} e^{-2|x|}\,|f(x)|$$

for $f\in C$ and $n=1,2,\dots$ satisfies the relations

$$p_n(Fy)\le \tfrac{1}{2}p_n(y)$$

for all $n=1,2,\dots$ and $y\in C$, proving thus that F is a generalized contraction.

Suppose now that our operator $F:C\to C$ is a topological contraction. As the constant $0\in C$ is the fixed point of F it follows that F would satisfy the conditions of our Lemma for some metric ρ inducing the topology of C. Let $U(0)$ be the neighbourhood of $\{0\}$ in C existing according to the Lemma and consider the fundamental system of neighbourhoods $\{U(n,a)\mid n\ge 1, a>0\}$ of $\{0\}$ defined by

$$U(n,a)=\{f\in C:p_n(f)<a\}.$$

It follows that there is some $n\ge 1$ and $a>0$ such that $U(n,a)\subset U(0)$ so that the neighbourhood $U(n,a)$ also would satisfy the conclusion of our Lemma. Choosing $V(0)$ to be $U(n+1,1)$ we consider the function $y_n\in C$ defined by $y_n(x)=0$ for $x\le n$ and $y_n(x)=x-n$ for $x>n$. Then obviously $b\cdot y_n\in U(n,a)$ for any constant b but on the other hand for every $x>n$ and any $k\ge 1$ we have $F^k y_n(x)>0$. Thus for any k we can choose b_k in such a way that

$$b_k F^k y_n(n+1)\cdot e^{-2(n+1)}\ge 1$$

showing that the sets $F^k(U(n, a))$ do not shrink into the set $U(n+1, 1)$ as would follow from the Lemma and the contradiction thus obtained completes the proof of our theorem.

Remark. If X is a metrizable topological space and $F: X \to X$ a continuous selfmapping then the sufficient and necessary conditions for F to be a topological contraction have been found by Ph. Meyers ([3]). It is an open problem to establish a similar characterization for generalized contractions dropping at the same time the hypothesis of metrizability of the space X. The question is:

Given a completely regular topological space X, how to characterize those continuous selfmappings $F: X \to X$ for which there exists a family $\{\rho_i \mid i \in I\}$ of pseudometrics ρ_i on X inducing the topology of X and a constant $q \in (0, 1)$ such that

$$\rho({}_iFx, Fy) \leq q\rho_i(x, y)$$

for all $x, y \in X$ and all $i \in I$?

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STABILITY OF CERTAIN DYNAMICAL SYSTEMS

BY

CLARK JEFFRIES

Introduction. For n real variables $\{x_i\}$ of time t we consider the following equations:

$$dx_i/dt = \sum_{j=1}^{n} a_{ij}(x, t) f_j(x_j) \tag{1}$$

Throughout $\{a_{ij}\}$ and $\{f_i\}$ are assumed to be continuous. For any given initial conditions there is assumed to be at least one trajectory solution. We assume $f_j(0)=0$ (so $\mathbb{O}=(0, 0, \ldots, 0)$ is a critical state), $f_j(u)$ has the same sign as u otherwise, and $\lim_{v\to\pm\infty} \int_0^v f_j(u)\, du = +\infty$. Two cases which will be mentioned below are $f_j(u)=u$ and $f_j(u)=\exp(u)-1$. The purpose of this note is to develop criteria for the (neutral or asymptotic) stability of (1) at $\mathbb{O}$.

Result. We will prove the following

THEOREM. *Suppose*

(i) *for each* x, t, i *that* $a_{ii}(x, t) \le 0$,

(ii) *for each* x, t, $i \neq j$ *that either* $a_{ij}(x, t) = a_{ji}(x, t) = 0$, *or* $a_{ij}(x, t) \neq 0$, $a_{ji}(x, t) \neq 0$, *and* $a_{ij}(x, t)/a_{ji}(x, t)$ *is a negative constant independent of* x, t, *and*

(iii) *for each* x, t *and each set of three or more distinct indices* $i, j, k, \ldots, p$ *that* $|a_{ij}, a_{jk}, \ldots, a_{pi}| = |a_{ip}, \ldots, a_{kj}, a_{ji}|$.

Then $\mathbb{O}$ *is a stable critical point for* (1), *and all trajectories for* (1) *may be bounded in terms of initial conditions for* x.

Proof. Define n positive real numbers $\{\lambda_i\}$ as follows. Let $\{i, j, \ldots\}$ be the set of indices of variables which interact with x_1. (A variable x_i interacts with x_1 if there is some x, t such that $a_{i1}(x, t) \neq 0$, or alternatively, $a_{1i}(x, t) \neq 0$.) If the set of such indices is empty, let $\lambda_1 = 1$ and start over with x_2. Otherwise, for each such index i let $\lambda_1 = 1$, $\lambda_i = -\lambda_1 a_{ji}/a_{i1}$, where for each i we choose x, t so that $a_{i1}(x, t) \neq 0$. Using condition (ii) each such λ_i is constant and we abbreviate by $\lambda_i = -\lambda_1 a_{1i}/a_{i1}$. Next consider the (possibly empty) set of all indices $\{k, l, \ldots\}$ of variables which interact with at least one of $\{x_i, x_j, \ldots\}$. Define for new indices $\{\lambda_k, \lambda_l, \ldots\}$ by $\lambda_k = -\lambda_i a_{ik}/a_{ki}$. Condition (iii) implies that this scheme is consistent. For suppose x_2 and x_3 interact with x_1 and with each other. Then we must have $\lambda_2 = -\lambda_1 a_{12}/a_{21}$

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and $\lambda_3=-\lambda_1 a_{13}/a_{31}$. Also we have $\lambda_3=-\lambda_2 a_{23}/a_{32}$, precisely because of condition (iii). So it goes for all loops of three or more interacting variables. That is, this scheme may be extended to those variables which interact with $\{x_k, x_l, \ldots\}$, and so on. When a maximal set of variables containing x_1 is so obtained, start over with some other variable, if any, and so on. The final result is a set of positive numbers $\{\lambda_i\}$ such that if $i\neq j$ and x_i and x_j interact, then $\lambda_i a_{ij}=-\lambda_j a_{ji}$ for x, t such that $a_{ij}\neq 0$, $a_{ji}\neq 0$, and so for all x, t. If $i\neq j$ and x_i and x_j do not interact, then $\lambda_i a_{ij}=-\lambda_j a_{ji}$ nonetheless. So for all $i\neq j$, x, t we have $\lambda_i a_{ij}=-\lambda_j a_{ji}$, that is, $\lambda_i^{1/2} a_{ij}\lambda_j^{-1/2}=-\lambda_j^{1/2} a_{ji}\lambda_i^{-1/2}$. To summarize this paragraph we may say that the conditions are sufficient to guarantee the existence of n positive constants $\{\lambda_i\}$ such that the similarity transformation on $\{a_{ij}\}$ by the diagonal matrix with $\lambda_i^{1/2}$ in the i,i-entry renders $\{a_{ij}\}$ into skew form plus non-positive diagonal entries for each x, t.

Next define a Liapunov function $\varphi(x)$ as

$$\varphi(x)=\sum_{i=1}\lambda_i\int_0^{x_i} f_i(u)\,du$$

Clearly $\varphi(x)>0$ unless $x=\mathbb{O}$. Clearly $\varphi(x)$ strictly increases without bound on all radial lines in n-space away from $\mathbb{O}$. Finally,

$$\begin{aligned} d\varphi/dt &= \sum_{i=1}^{n}\lambda_i f_i(x_i)\,dx_i/dt \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}[\lambda_i^{1/2} f_i(x_i)]\lambda_i^{1/2} a_{ij}(x,t)\lambda_j^{-1/2}[\lambda_j^{1/2} f_j(x_j)] \\ &= \sum_{i=1}^{n} a_{ii}(x,t)\lambda_i f_i(x_i)^2 \leq 0 \end{aligned}$$

so $\varphi(x)$ is non-increasing along all trajectories. The level sets of $\varphi(x)$ are homeomorphic to $(n-1)$-spheres centered about $\mathbb{O}$. Hence any trajectory for (1) may be bounded by the initial state x in n-space. Q.E.D.

DISCUSSION. The case $f_j(u)=u$ is particularly simple and of mathematical interest. However, if $f_j(u)=N_j(e^u-1)$, $N_j>0$, then the theorem is relevant to theoretical ecology. Making the change of variables $y_i=N_i \exp x_i$ we have

$$dy_i/dt=\sum_{j=1}^{n} y_i\tilde{a}_{ij}(y,t)(y_j-N_j) \tag{2}$$

where $\tilde{a}_{ij}(y,t)=a_{ij}(x,t)$. This is a direct generalization of the Lotka-Volterra equations with critical state $(N_1, N_2, \ldots, N_n)$ [1]. Notice that $\{\tilde{a}_{ij}\}$ satisfies conditions (i), (ii), (iii) if and only if $\{a_{ij}\}$ satisfies the same conditions. Thus if $\{\tilde{a}_{ij}\}$ satisfies the conditions, $(N_1, N_2, \ldots, N_n)$ must be a stable critical state for (2). We state the above as a

COROLLARY. *If* $\tilde{a}_{ij}$ *satisfies conditions* (i), (ii), (iii), *then* $(N_1, N_2, \ldots, N_n)$ *is a stable critical state for* (2).

The corollary may be interpreted as follows in mathematical ecology. Suppose the dynamics of an ecosystem may be expressed in the form (2), fulfilling the conditions. Condition (i) means that intraspecific interactions are of the self-regulating type. Condition (ii) means that for each pair of species and each x, t there is either no interaction or an interaction of the predator-prey or parasite-host type. It is possible with condition (ii) that two species might interact only at certain x, t, say, only in the absence of a preferred prey or only at certain times of the year. Condition (ii) would not allow, however, one species to affect a second species without the second species affecting the first. For example, condition (ii) would not allow a predator to kill its prey without benefiting from the kill. Should, say, a preferred prey become abundant, the effects of a given predator and given prey per animal on each other would both be reduced. Condition (ii) requires that such reductions be proportional.

Condition (iii) means that if for some x, t loops exist in the food web of the ecosystem, then the loops must be in a sense balanced. For example, suppose zooplankton are preyed upon by both cod and capelin and that cod prey upon capelin. This constitutes a loop in the Arctic Ocean food web. If the system may be modeled as (2), then for stability we might expect the absolute value of the products of the interaction coefficients of the two cycles to be equal.

It is important to note that such equality is invariant of rescaling. Given (2) suppose we define new variables $z_i=\gamma_i y_i$, $\gamma_i>0$. Then we have

$$dz_i/dt = \sum_{j=1}^{n} z_i \hat{a}_{ij}(z, t)(z_j - \hat{N}_j) \tag{3}$$

where the new coordinates of the critical state are $\hat{N}_j=\gamma_j N_j$ and the new interaction matix is $\hat{a}_{ij}(z, t)=\tilde{a}_{ij}(y, t)\gamma_j^{-1}$. Note that $\{\hat{a}_{ij}\}$ fulfills the conditions if and only if $\{\tilde{a}_{ij}\}$ fulfills the conditions.

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FUNCTIONS BELONGING TO A DIRICHLET SUBALGEBRA OF THE DISK ALGEBRA

BY
BRUCE LUND

Browder and Wermer in [2] give a method for constructing Dirichlet subalgebras of the disk algebra. In this note we show that these Dirichlet algebras do not contain any non-constant functions which satisfy a Lipschitz-one condition on a subinterval of the unit circle.

Let A be a uniform algebra on a compact Hausdorff space X (see [6] for facts about uniform algebras). We say that A is a *Dirichlet algebra on X* if $\operatorname{Re} A = \{\operatorname{Re}(f): f \in A\}$ is uniformly dense in $C_R(X)$, the real-valued continuous functions on X. Let $T=\{z:|z|=1\}$ and $U=\{z:|z|<1\}$ be the unit circle and the open unit disk in the complex plane. The *disk algebra*, $A(T)=\{f \in C(T): f$ extends analytically to $U\}$ is a Dirichlet algebra on T (see, for example, [4], p. 43). Browder and Wermer in [2] give a method for constructing subalgebras of $A(T)$ which are still Dirichlet algebras on T. Their method goes as follows: Let $p(e^{it})$ be a homeomorphism of T such that $dp(e^{it})/dt=0$ a.e. with respect to Lebesgue measure on T. Define

$$A_p = A(T) \cap \{f \in C(T): f \circ p \in A(T)\}.$$

Browder and Wermer show that A_p is a Dirichlet algebra on T. In [1] Blumenthal shows that A_p is a maximal uniform algebra in $A(T)$.

The method of showing that A_p is a Dirichlet algebra is indirect in that the result is obtained by showing that there are no non-zero real annihilating measures for A_p. No work appears to have been done on describing what types of functions may belong to A_p. Our theorem below gives a result in this direction.

THEOREM. *If $f \in A_p$ and if f satisfies a Lipschitz-one condition on some subinterval of T, then f is a constant function.*

This theorem is a consequence of the following lemma which is of some independent interest.

LEMMA. *Suppose $F \in A(T)$ and suppose there is an interval $I=\{\exp(it): a \le t \le b\}$ so that F is of bounded variation on I. Assume also that*

$$\frac{dF(e^{it})}{dt} = 0 \quad \text{a.e.} \quad \text{for} \quad t \in [a, b].$$

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Then F is a constant function.

Proof. We first note that if $I=T$, then the Lemma follows directly from the following two classical results which may be found in [3], p. 42.

(1) If $f \in A(T)$ and if $f(e^{it})$ is of bounded variation on T, then $f(e^{it})$ is absolutely continuous on T.

(2) If $f \in A(T)$, then $f(e^{it})$ is absolutely continuous on T if and only if $f'(z) \in H^1(U)$. Moreover, if $f'(z) \in H^1(U)$, then

$$\frac{df(e^{it})}{dt} = ie^{it} \lim_{r\to 1} f'(re^{it}) \quad \text{a.e.}$$

For the case $I \neq T$, the idea of the proof is as follows. We multiply F by a suitable polynomial in order to form a new function G which is of bounded variation along a simple closed piecewise analytic curve in $\bar{U}$ which contains I. Statements (1) and (2) can be applied to the function $G \circ \phi(z)$ where $\phi(z)$ is a conformal map of U onto the region lying inside the curve. The final result then follows by a simple argument.

Let $F(z)$ denote the analytic extension of $F(e^{it})$ to U and let $M=\max\{|F(z)|: |z|\leq 1\}$. An elementary computation involving the Cauchy integral formula gives

$$|F'(re^{it})| \leq \frac{M}{1-r} \quad \text{for} \quad 0 \leq r < 1 \tag{3}$$

Set

$$G(z) = F(z)[(1-e^{-ia}z)(1-e^{-ib}z)]^2$$

Since we have $G'(\mathrm{r}e^{it})=(\partial G(\mathrm{r}e^{it})/\partial r)e^{-it}$ for $0\leq r<1$, we may use (3) to conclude that $(\partial G(\mathrm{r}e^{ia})/\partial r)$ and $(\partial G(\mathrm{r}e^{ib})/\partial r)$ are continuous for $0\leq r\leq 1$.

We let $V=\{\mathrm{r}e^{it}: a<t<b \text{ and } 0<r<1\}$. Suppose $\phi(z)$ is a conformal map from U onto V. Then ϕ extends to be a homeomorphism of $\bar{U}$ onto $\bar{V}$, and $\phi(e^{it})$ is absolutely continuous ([3], p. 44). We will suppose that $\phi(1)=e^{ia}$, $\phi(e^{it_1})=e^{ib}$, and $\phi(e^{it_2})=0$.

The hypothesis that $F(e^{it})$ is of bounded variation along I implies that $G \circ \phi(e^{it})$ is of bounded variation on $[0, t_1]$. Moreover, the continuity of $(\partial G(\mathrm{r}e^{ia})/\partial r)$ and $(\partial G(\mathrm{r}e^{ib})/\partial r)$ for $0\leq r\leq 1$ implies that $G \circ \phi(e^{it})$ is of bounded variation on $[t_1, 2\pi]$. Consequently, $G \circ \phi(e^{it})$ is of bounded variation on T. Since $G \circ \phi(z) \in A(T)$, we can conclude by (1) and (2) that $(d(G \circ \phi(e^{it}))/dt)$ gives the boundary values of a function in $H^1(U)$.

By hypothesis $(dF(e^{it})/dt)=0$ a.e. for $t \in [a, b]$ and from this we obtain $d(F \circ \phi(e^{it})/dt)=0$ a.e. for $t \in [0, t_1]$ ([5], Corollary 2). If we let

$$P(z) = [(1-e^{-ia}z)(1-e^{-ib}z)]^2,$$

then

$$\frac{d(G \circ \phi(e^{it}))}{dt} = \frac{d(F \circ \phi(e^{it}))}{dt} P \circ \phi(e^{it}) + F \circ \phi(e^{it}) \frac{d(P \circ \phi(e^{it}))}{dt}$$

a.e. for $t \in [0, 2\pi]$. But then

$$\frac{d(G \circ \phi(e^{it}))}{dt} = F \circ \phi(e^{it}) \frac{d(P \circ \phi(e^{it}))}{dt} \quad \text{a.e.} \quad \text{for} \quad t \in [0, t_1].$$

Since $F \circ \phi(e^{it})(d(P \circ \phi(e^{it}))/dt)$ gives the boundary values for a function in $H^1(U)$, we therefore conclude that $(d(F \circ \phi(e^{it}))/dt)=0$ a.e. for $t \in [0, 2\pi]$. Since $F \circ \phi(z)$ extends analytically across $\{e^{it}: t_1 < t < t_2\}$, it follows that $F(z)$ is a constant function. This completes the proof.

Proof of Theorem. If $f(e^{it})$ satisfies the hypothesis of the theorem, then $F(e^{it}) = f \circ p(e^{it}) \in A(T)$. If $f(e^{it})$ is Lip-1 on $\{e^{it}: A \leq t \leq B\}$, then $F(e^{it})$ is of bounded variation on $p^{-1}(\{e^{it}: A \leq t \leq B\}) = \{e^{it}: a \leq t \leq b\}$. The singularity of $p(e^{it})$ and the Lip-1 condition on $f(e^{it})$ imply that $dF(e^{it})/dt)=0$ a.e. for $t \in [a, b]$. We now apply the Lemma to conclude that F, and hence f, is identically constant on T.

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UNIQUENESS OF GENERALIZED SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

BY
M. A. MALIK

1. Let Ω be an open subset of R and H be a complex Hilbert space; (,) represents scalar product in H. Let also A be a closed linear operator with domain D_A dense in H and A^* with domain D_A^* be its adjoint. Under graph scalar product D_A and D_A^* are also Hilbert spaces. By $\mathscr{D}_\Omega(H)$ we denote the space of all infinitely differentiable functions (H-valued) with compact support defined on Ω. $\mathscr{D}_\Omega(H)$ is equipped with Schwartz topology. Similarly, we define $\mathscr{D}_\Omega(D_A)$, $\mathscr{D}_\Omega(D_A^*)$ and $\mathscr{D}_\Omega(\mathbb{C})$; $\mathbb{C}$ represents the complex plane. By $\mathscr{D}'_\Omega(H)=\mathscr{L}(\mathscr{D}_\Omega(\mathbb{C}); H)$ we mean the space of all continuous linear mappings (H-valued) defined on $\mathscr{D}_\Omega(\mathbb{C})$. In a similar way, we define $\mathscr{D}'_\Omega(D_A)$. For $\Psi\in\mathscr{D}_\Omega(\mathbb{C})$ and $u\in\mathscr{D}'_\Omega(H)$, $\langle u,\Psi\rangle\in H$. It is easy to show that if $\langle u,\Psi\rangle\in D_A$ for all $\Psi\in\mathscr{D}_\Omega(\mathbb{C})$, then $u\in\mathscr{D}'_\Omega(D_A)$. The H-valued distribution space $\mathscr{D}'_\Omega(H)$ is also the dual of $\mathscr{D}_\Omega(H)$. In this case, for $\varphi\in\mathscr{D}_\Omega(H)$ and $u\in\mathscr{D}'_\Omega(H)$, $\langle u,\Psi\rangle\in\mathbb{C}$.

We define Au, for $u\in\mathscr{D}'_R(D_A)$ by the relation

$$\langle Au,\Psi\rangle = A\langle u,\Psi\rangle \tag{1.1}$$

for all $\Psi\in\mathscr{D}_R(\mathbb{C})$; $Au\in\mathscr{D}'_R(H)$.

For convenience, we write $L=(1/i)(d/dt)-A$ and $L^*=(1/i)(d/dt)-A^*$. By $R(\lambda; A)$, we denote the resolvent operator of A, $\lambda\in\mathbb{C}$. In view of imposing condition on A, we need:

DEFINITION. Let $\mathscr{F}$ be a family of parallel lines $\{\operatorname{Im}\lambda=\tau_n,\ \tau_n\to\infty \text{ as } n\to\infty,\ \tau_n\to-\infty \text{ as } n\to-\infty\}$ in the complex plane $\mathbb{C}$. Let r be a positive real number and j, m be positive integers. We shall say that the resolvent $R(\lambda; A)$ is of (j, r, m)-growth on $\mathscr{F}$ if $R(\lambda; A)$ exists for λ outside j intervals of length r on every line of $\mathscr{F}$ and for these λ

$$|R(\lambda; A)| \le \text{const. } |\lambda|^m \tag{1.2}$$

Throughout this paper, the 'const.' need not be the same constant.

2. We consider the abstract differential equation

$$\frac{1}{i}\frac{du}{dt}-Au=f \tag{2.1}$$

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The author has recently proved the existence of weak solution of the equation (2.1) imposing condition on the resolvent $R(\lambda; A^*)$ in [2]. In fact, he proved:

THEOREM A. *If $R(\lambda; A^*)$ is of (j, r, m)-growth on $\mathscr{F}$, then for every $f \in \mathscr{D}'_A(H)$ the equation $Lu=f$ has at least one weak solution $u \in \mathscr{D}'_R(H)$, i.e.,*

$$\langle u, L^*\varphi\rangle = \langle f, \varphi\rangle \tag{2.2}$$

for all $\varphi \in \mathscr{D}(D'_R)$.

In this paper, we show that the solution u in Theorem A is not unique and actually $u \in \mathscr{D}'(D_A)$ yielding a solution of (2.1). We also study the uniqueness of the solution u (of Lu=0) vanishing in a neighbourhood of $x \in R$.

3. We prove:

THEOREM 1. *If $R(\lambda; A^*)$ is of (j, r, m)-growth on $\mathscr{F}$, the space of weak solutions of $Lu=0$ consists of more than one element.*

In the proof of Theorem 1, we need the following:

DEFINITION. We define V_Ω as the set of all $u \in \mathscr{D}'_\Omega(H)$ such that

$$\langle u, L^*\varphi\rangle = 0 \tag{3.1}$$

for all $\varphi \in \mathscr{D}_\Omega(D^*_A)$.

LEMMA 1. *Let the hypothesis of Theorem 1 be satisfied and Ω_1, Ω_2 be two open subsets of R with $\Omega_1 \subset \Omega_2$. Then V_R is dense in V_{Ω_2} under the topology of $\mathscr{D}'_{\Omega_1}(H)$, i.e., for $\varphi \in \mathscr{D}_{\Omega_1}(H)$ if $\langle \chi, \varphi\rangle=0$ for all $\chi \in V_R$ then $\langle \mu, \varphi\rangle=0$ for all $\mu \in V_{\Omega_2}$.*

S. Zaidman [4] has proved a similar result for $L^2_{\mathrm{loc}}(H)$, the space of locally square integrable H-valued functions and Lemma 1 can be proved along the same line.

From Lemma 1, we immediately have:

LEMMA 2. *Under the hypothesis of Theorem 1, if $V_R=\{0\}$ then for any $\Omega \subset R$, $V_\Omega=\{0\}$.*

Proof of Theorem 1. Suppose on contrary, that $V_R=\{0\}$. In such a case, we shall show that the weak solution of $Lu=\delta\otimes x$ does not exist i.e., there exists no $u \in \mathscr{D}'_R(H)$ satisfying

$$\langle u, L^*\varphi\rangle = \langle \delta\otimes x, \varphi\rangle \tag{3.2}$$

for all $\varphi \in \mathscr{D}(D^*_A)$. As it contradicts Theorem A the proof will be complete.

Now suppose there exists $u \in \mathscr{D}'_R(H)$ satisfying (3.2). For $\varphi \in \mathscr{D}(D^*_A)$ with supp $\varphi \subset (0, \infty)$ we obviously have $\varphi(0)=0$ and so

$$\langle u, L^*\varphi\rangle = 0 \tag{3.3}$$

since $V_R=\{0\}$, from Lemma 2, $u=0$ on $(0, \infty)$; let $\Omega=(0, \infty)$. Similarly, $u=0$ on $(-\infty, 0)$. So the supp u is concentrated at the origin and u may therefore be expressed as a finite linear combination of Dirac distribution and its derivative, hence:

$$u = \sum_{0}^{n} a_k \otimes \delta^{(k)} \tag{3.4}$$

$a_k \in H$. Substituting (3.4) in (3.2) and using (1.1) after transposing the derivative, we have

$$\sum_{0}^{n} (a_k, (-i)^{k+1}\varphi^{(k+1)}(0) - A^*\varphi^{(k)}(0)) = (x, \varphi(0)) \tag{3.5}$$

for all $\varphi \in \mathscr{D}_R(D_A^*)$ and $\chi \in H$. A choice of φ in (3.5) such that $\varphi^{(k)}(0)=0$ for $k=0, 1, 2, \ldots, n$ whereas $\varphi^{(n+1)}(0)\neq 0$ implies the leading coefficient $a_n=0$. Thus $u\equiv 0$. It contradicts (3.2). This completes the proof.

Theorem 2. *Let $R(\lambda; A^*)$ be of (j, r, m)-growth on $\mathscr{F}$. Then for any $f \in \mathscr{D}'_R(H)$, the abstract differential equation $Lu=f$ has more than one solution $u \in \mathscr{D}'_R(D_A)$.*

Proof of Theorem 2. From Theorem 1, there exists more than one $u \in \mathscr{D}'_R(H)$ such that

$$\langle u, L^*\varphi\rangle = \langle f, \varphi\rangle. \tag{3.6}$$

We shall show that $u \in \mathscr{D}'_R(D_A)$. Putting $\varphi=\Psi\otimes x$, $\Psi \in \mathscr{D}_R(\mathbb{C})$ and $\chi \in D_A^*$ in (3.6) we have

$$\left\langle u, \frac{1}{i}\frac{d}{dt}\Psi\otimes x - A^*\Psi\otimes x\right\rangle = \langle f, \Psi\otimes x\rangle \tag{3.7}$$

from where

$$\left(\left\langle \frac{1}{i}\frac{du}{dt} - f, \Psi\right\rangle, x\right) = (\langle u, \Psi\rangle, A^*x) \tag{3.8}$$

for all $\chi \in D_A^*$. This implies that $\langle u, \Psi\rangle \in D_A^{**} = D_A$ as A is a closed linear operator with domain D_A dense in H; (see [3], pages 196–197). Consequently, $u \in \mathscr{D}'(D_A)$ and satisfies the equation

$$\frac{1}{i}\frac{du}{dt} - Au = f.$$

Theorem 3. *Let $u \in \mathscr{D}'_R(D_A)$ be a solution of*

$$\frac{1}{i}\frac{du}{dt} - Au = 0 \tag{3.9}$$

and the resolvent $R(\lambda; A)$ is of (j, r, m)-growth on $\mathscr{F}$. If for some $\chi \in R$ and $\varepsilon>0$, u vanishes on $(x-\varepsilon, x+\varepsilon)$, then $u\equiv 0$.

LEMMA 3. [2] *Let* $R(\lambda; A)$ *be of* (j, r, m)*-growth on* $\mathscr{F}$ *and* $\xi \in C^\infty(D_A)$ *be a solution of* $L\varphi=0$ *on* $a \leq t \leq b$ *with* $\xi(c)=0$, $a<c<b$. *Then* $\xi \equiv 0$ *on* $[a, b]$.

Proof of Theorem 3. Consider a sequence $\{\alpha_n; \alpha_n \in \mathscr{D}_R(\mathbb{C}), \operatorname{supp} \alpha_n \subset [-1/n, 1/n]\}$ such that $\alpha_n \to \delta$, the Dirac distribution. Let $u \in \mathscr{D}'_R(D_A)$ be a solution of (3.9). Consider the convolution $u * \alpha_n$. It is clear that $u * \alpha_n \in C^\infty(D_A)$, $L(u * \alpha)=0$ and for sufficiently large n,

$$\operatorname{supp}(u * \alpha_n) \cap \left(x-\varepsilon+\frac{1}{n}, x+\varepsilon-\frac{1}{n}\right) = \phi$$

so $(u * \alpha_n)(x)=0$. In view of Lemma 3, $u * \alpha_n=0$ on any interval $a \leq t \leq b$ containing x and so $(u * \alpha_n)(t)=0$ for all $t \in R$. Consequently $u \equiv 0$.

4. Finally, we present the following version of an example of S. Agmon and L. Nirenberg [1] where the conclusion like of Lemma 2 is not true.

EXAMPLE. In the space of all continuous complex functions defined on R, consider a closed linear operator $A=i(d/dx)$ with domain D_A consisting of all C^1 functions vanishing at $-\infty$. Consider the homogeneous equation

$$\frac{1}{i}\frac{\partial}{\partial t}u - i\frac{\partial}{\partial x}u = 0 \tag{4.1}$$

on $t_1 \leq t \leq t_2$, $u(t, \cdot) \in D_A$. The operator $-iL=(1/i)(\partial/\partial t)-i(\partial/\partial x)$ is a directional derivative in the (t, x)-plane. Any solution u of (4.1) is constant on the line with direction (1.1) lying in the strip $[t_1, t_2] \times R$ and need not be zero. However, if $t_1=-\infty$ and $u(t, x)$ is a solution of (4.1), then $u \equiv 0$; in fact, u is constant on the line $x=t+c$ and vanishes at $x=-\infty$.

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INNER FUNCTIONS AND ISOMETRIES

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Our objective in this note is to prove a theorem about the spectral behavior of certain multiplication operators. It was motivated by and yields an extension of part of a theorem of Lumer [3] concerning the possibility of imbedding the classical Hardy spaces in abstract ones.

We fix a σ-finite measure m on a measure space X and we regard $L^\infty(m)$ either as a space of functions or as the von Neumann algebra of multiplication operators on $L^2(m)$. Also, we fix a (closed) subspace $\mathfrak{M}$ of $L^2(m)$ and let $\mathfrak{U}$ denote the algebra of all $\phi \in L^\infty(m)$ such that $\varphi\mathfrak{M} \subseteq \mathfrak{M}$. A uni-modular function in $\mathfrak{U}$ will be called an *inner function*.

THEOREM. *Suppose that* (*a*) *there is a function f in $\mathfrak{M}$ which vanishes at most on a null set such that the linear manifold $\{\phi f \mid \phi \in \mathfrak{U}\}$ is dense in $\mathfrak{M}$ and* (*b*) *the only real-valued functions in $\mathfrak{U}$ are constants. Then for each nonconstant inner function θ in $\mathfrak{U}$, the spectral measure of θ regarded as a unitary operator on $L^2(m)$ is mutually absolutely continuous with respect to Lebesgue measure on the unit circle.*

The result of Lumer is contained in the following corollary whose proof we omit because it is easily constructed from the Theorem and straightforward arguments in spectral theory of the sort found in Chapitre III no 2 of [4]. Here, $L^\infty(\mathbf{T})$ and H^∞ $(\mathbf{T})$ are the usual Lebesgue and Hardy spaces on the unit circle $\mathbf{T}$.

COROLLARY. *Suppose that hypotheses* (a) *and* (b) *of the Theorem are satisfied and that θ is a nonconstant inner function in $\mathfrak{U}$. For each polynomial $p(z, \bar{z})$ in z and $\bar{z}$ on $\mathbf{T}$, let $p(\theta, \bar{\theta})$ be the same polynomial in θ and $\bar{\theta}$ on X. Then the correspondence $p(z, \bar{z}) \to p(\theta, \bar{\theta})$ extends to be an isometric $*$-isomorphism from $L^\infty(\mathbf{T})$ into $L^\infty(m)$ which is also continuous with respect to the weak-$*$ topologies on $L^\infty(\mathbf{T})$ and $L^\infty(m)$ and which carries $H^\infty(\mathbf{T})$ into $\mathfrak{U}$.*

To see that Lumer's result is contained in the Corollary, momentarily let X be a compact Hausdorff space, let A be a function algebra on X and let m be a representing measure for a point in the maximal ideal space of A. Then if $\mathfrak{M} = H^2(m)$, the closure of A in $L^2(m)$, $\mathfrak{U}$ is $H^\infty(m)$, the intersection $L^\infty(m) \cap H^2(m)$. It is clear that $\mathfrak{M}$ satisfies hypothesis (a) with f taken to be the constant function 1; and it is well-known that since m is multiplicative on A, $\mathfrak{M}$ satisfies hypothesis (b) as well.

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Hence, given a nonconstant inner function, the Corollary shows how to use it to imbed $L^\infty(\mathbf{T})$ and $H^\infty(\mathbf{T})$ into $L^\infty(m)$ and $H^\infty(m)$, resp.; and this is part of the result of Lumer. He also shows how to imbed the other L^p and H^p spaces on the circle into the corresponding ones on X and our results yield this part of his theorem too; but we shall not enter into the details.

The proof of the Theorem is based on two preparatory lemmas of independent interest.

If φ lies in $\mathfrak{U}$, then φ_+ will denote the restriction $\varphi \mid \mathfrak{M}$ of φ to $\mathfrak{M}$ and $\mathfrak{U}_+$ will denote the algebra of all such restrictions.

LEMMA 1. *If* $\mathfrak{M}$ *satisfies hypothesis* (a) *of the Theorem, then* $\mathfrak{U}_+$ *is a maximal abelian algebra of operators on* $\mathfrak{M}$. *If, in addition,* $\mathfrak{M}$ *satisfies hypothesis* (b), *then* $\mathfrak{U}_+$ *is irreducible.*

Proof. Let f be a function in $\mathfrak{M}$ satisfying the conditions of hypothesis (a) and let $d\mu=|f|^2\,dm$. Then μ is a finite measure and the map U defined by the formula $Ug=g/f$, $g \in L^2(m)$, is a Hilbert space isomorphism of $L^2(m)$ onto $L^2(\mu)$ which carries f to the constant function 1. It is clear that $UL^\infty(m)U^{-1}=L^\infty(\mu)$ and that if $\mathfrak{M}'=U\mathfrak{M}$, then $U\mathfrak{U}U^{-1}$ is the set of all $\varphi \in L^\infty(\mu)$ such that $\varphi\mathfrak{M}' \subseteq \mathfrak{M}'$. Likewise it is clear that $\mathfrak{M}'$ satisfies hypothesis (a) with $\mathfrak{U}$ replaced by $U\mathfrak{U}U^{-1}$ and f replaced by 1. Hence we may assume without loss of generality that m is finite and that $f=1$. In this case, hypothesis (a) implies that $\mathfrak{U}=L^\infty(m) \cap \mathfrak{M}$ and so if A is any operator on $\mathfrak{M}$ which commutes with $\mathfrak{U}_+$, then A must be the multiplication operator determined by the function $\varphi=A1$ in $\mathfrak{M}$. But then, standard arguments (see [2, pp. 212–213]) show that φ lies in $L^\infty(m)$ as well and thus $A=\varphi_+$ belongs to $\mathfrak{U}_+$.

Suppose $\mathfrak{M}$ also satisfies hypothesis (b) and that P is the projection of $\mathfrak{M}$ onto a subspace which reduces $\mathfrak{U}_+$. Then P commutes $\mathfrak{U}_+$ and so by what was just proved $P=\varphi_+$ for some φ in $\mathfrak{U}$. Since P is idempotent, φ takes on only the values zero or one a.e.(m). Hence, by hypothesis (b), φ is constant and so P is either the identity or the zero operator. Whence $\mathfrak{U}_+$ is irreducible and the proof is complete.

Suppose U is a unitary operator on a Hilbert space $\mathscr{L}$ and let $E(\cdot)$ be its spectral measure. If $\mathscr{L}_{AC}$ (resp., $\mathscr{L}_S$) is the set of all f in $\mathscr{L}$ such that the scalar measure $(E(\cdot)f,f)$ is absolutely continuous (resp., singular) with respect to Lebesgue measure on $\mathbf{T}$, then, as is well-known, $\mathscr{L}_{AC}$ and $\mathscr{L}_S$ are orthogonal supplementary spectral subspaces of U and so reduce every operator commuting with U. The space $\mathscr{L}_{AC}$ (resp., $\mathscr{L}_S$) is called the absolutely continuous (resp., singular) spectral subspace of U and U is called absolutely continuous (resp., singular) in case $\mathscr{L}_{AC}=\mathscr{L}$ (resp., $\mathscr{L}_S=\mathscr{L}$). Suppose now that V is an isometry on a Hilbert space $\mathscr{H}$ and let W be its minimal unitary extension acting on the Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. Also, let $V=(V \mid \mathscr{E})\oplus(V \mid \mathscr{T})$ with $\mathscr{H}=\mathscr{E}\oplus\mathscr{T}$ be the Wold decomposition of V so that $V \mid \mathscr{E}$ is a pure isometry on $\mathscr{E}$ and $V \mid \mathscr{T}$ is a unitary operator $\mathscr{T}$ (see [2, p. 74]). Since the minimal unitary extension of a pure isometry is a bilateral shift (of suitable multiplicity) and since these are absolutely continuous, it follows that the singular spectral subspace of $V|\mathscr{T}$ on $\mathscr{T}$ coincides with the singular spectral subspace of W on $\mathscr{K}$. So, in particular, W is absolutely continuous if and only if $V|\mathscr{T}$ is. Finally, note that the results of [1] show that every operator on $\mathscr{H}$ which commutes with V is reduced by the singular spectral subspace of $V|\mathscr{T}$.

LEMMA 2. *Let* $\mathfrak{M}$ *satisfy the hypotheses* (a) *and* (b) *of the Theorem and let* θ *be a nonconstant inner function in* $\mathfrak{A}$. *Then the isometry* θ_+ *on* $\mathfrak{M}$ *is non-unitary and* (*provided it has one*) *the unitary summand in its Wold decomposition is absolutely continuous.*

Proof. If θ_+ were unitary, then $\mathfrak{M}$ would reduce the unitary multiplication operator θ on $L^2(m)$ and so θ and $\bar{\theta}$ would lie in $\mathfrak{A}$. The hypothesis (b) would then imply that θ is constant which is contrary to assumption. If the unitary summand in the Wold decomposition of θ_+ had a singular spectral subspace $\mathscr{E}_S$, then by the remarks in the preceding paragraph, $\mathscr{E}_S$ would reduce $\mathfrak{A}_+$. By Lemma 1, then, $\mathscr{E}_S$ must be either all of $\mathfrak{M}$ or the zero subspace. Since the first case is ruled out by the fact that θ_+ is not unitary, $\mathscr{E}_S=\{0\}$ and the proof is complete.

Proof of the Theorem. As in the proof of Lemma 1, we may assume without loss of generality that the constant function 1 lies in $\mathfrak{M}$. Since θ is unitary and lies in the abelian von Neumann algebra $L^\infty(m)$, its spectral projections lie in $L^\infty(m)$ also and so must be of the form $\mathfrak{x}_M$ for measurable sets M. Let $\mathfrak{x}_{M_S}$ be the spectral projection onto the singular spectral subspace of θ and let $\mathscr{K}$ be the span of the spaces $\bar{\theta}^n\mathfrak{M}$, $n=0, 1, 2, \ldots$. Then $\mathscr{K}$ reduces θ and $\theta|\mathscr{K}$ is the minimal unitary extension of θ_+ on $\mathfrak{M}$. By Lemma 2 and the remarks in the paragraph preceding it, $\theta|\mathscr{K}$ is absolutely continuous and, therefore, $\mathscr{K}$ and $\mathfrak{x}_{M_S}L^2(m)$ are orthogonal. But then, since 1 lies in $\mathfrak{M}$ which in turn is contained in $\mathscr{K}$, this implies that $m(M_S)=(\mathfrak{x}_{M_S}, 1)=0$. Thus the spectral measure $E(\cdot)$ of θ is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$. To see that Lebesgue measure on $\mathbf{T}$ is absolutely continuous with respect to $E(\cdot)$ simply note that by Lemma 2, θ has a reducing subspace on which it is a bilateral shift. Since this implies that an $E(\cdot)$-null set is a Lebesgue null set, the proof is complete.

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INJECTIVITY AND EQUATIONAL COMPACTNESS IN THE CLASS OF $\aleph_0$-SEMILATTICES

BY
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This note presents characterizations of the injective and of the equationally compact $\aleph_0$-semilattices, which are analogous, respectively, to the characterizations of the injective semilattices given by Bruns and Lakser [2] and of the equationally compact semilattices given by Grätzer and Lakser [3].

It is known (see Banaschewski [1]) that in an equational class of finitary algebras, the existence of enough injectives guarantees the existence of injective hulls; it will be seen here that the class of $\aleph_0$-semilattices has enough injectives, while not every $\aleph_0$-semilattice has an injective hull.

As a sideline, we will see that the class of $\aleph_0$-semilattices also provides an example of an equational class of infinitary algebras in which Birkhoff's Subdirect Representation Theorem holds, but in which the Pure Representation Theorem (proved by Walter Taylor [5] for finitary algebras) does not hold.

1. **Preliminaries.** An $\aleph_0$-semilattice is a semilattice S with the usual binary operation $\wedge$, and an additional $\aleph_0$-ary operation $\bigwedge$ which associates with each $\sigma \in S_\omega$ (ω the set of natural numbers) the meet, $\bigwedge\sigma$, and which satisfies the obvious identities. For $\sigma \in S^\omega$, we sometimes write $\bigwedge_{n\in\omega} \sigma(n)$ or $\bigwedge\{\sigma(n) \mid n \in \omega\}$ for $\bigwedge \sigma$.

S is the class of all $\aleph_0$-semilattices. By "homomorphism" we will always mean mappings which preserve countable meets.

Each $\aleph_0$-semilattice S has a natural partial ordering on it given by $a \leq b$ iff $a \wedge b = a$; with respect to this partial ordering $a \wedge b$ is the greatest lower bound of a and b, and $\bigwedge\sigma$ is the greatest lower bound of $\{\sigma(n) \mid n \in \omega\}$ for $\sigma \in S^\omega$. The symbol "$\bigvee$" is used to denote join (least upper bound) with respect to this partial ordering, and the symbol "$\bigwedge$" is used indiscriminately to denote meet (least upper bound) of arbitrary (not necessarily countable) subsets.

It is well-known that, for any semilattice S, the map which assigns to each $s \in S$, $\{t \in S \mid t \leq s\}$, is a semilattice embedding of S into $P(S)$, the set of all subsets of S, with the operation of set-intersection, and that moreover this embedding takes arbitrary (existing) meets to the corresponding set intersection in $P(S)$. In particular, it follows that every $\aleph_0$-semilattice has an $\aleph_0$-embedding into a power of **2**, the

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two-element $\aleph_0$-semilattice. Consequently, Birkhoff's Subdirect Representation Theorem holds in **S**, and **2** is the only subdirectly irreducible member of **S**.

Recall that the *characteristic* of a (possibly infinitary) algebra is the smallest infinite regular cardinal greater than the arities of all the fundamental operations. If A and B are algebras (of the same type) of characteristic m, then a homomorphism $f:A\to B$ is a *pure embedding* (see Nelson [4]) iff, for all X, every subset of $A[X]^2$ with fewer than m elements is contained in the kernel of a homomorphism $A[X]\to A$ over A whenever it is contained in the kernel of a homomorphism $A[X]\to B$ over f (where $A[X]$ is the free extension of A by the set X, in any equational class containing both A and B). Since $\aleph_0$-semilattices are of characteristic $\aleph_1$, a homomorphism $f:S\to T$ in S is a pure embedding iff every at most countable subset of $S[X]^2$ ($S[X]$ the free extension of S in **S** by the set X) is contained in the kernel of a homomorphism $S[X]\to S$ over S whenever it is contained in the kernel of a homomorphism $S[X]\to T$ over f.

Let $\mathbf{N}\in\mathbf{S}$ have as underlying partially ordered set the natural numbers with the usual order; then the embedding $\phi:\mathbf{N}\to\mathbf{N}\times\mathbf{N}$ given by

$$\begin{aligned}\phi(0) &= (0,0)\\ \phi(1) &= (1,0)\\ \phi(n+1) &= (n,n) \quad\text{for}\quad n\geq 1\end{aligned}$$

has a retraction, namely $\psi:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ given by

$$\begin{aligned}\psi((0,n)) &= 0 \quad\text{for all}\quad n\\ \psi((n,0)) &= 1 \quad\text{for all}\quad n\geq 1\\ \psi((n,m)) &= (n\wedge m)+1 \quad\text{for}\quad n,m\geq 1,\end{aligned}$$

and thus in particular ϕ is a pure embedding. However, $p_1\phi$ and $p_2\phi$ (p_1,p_2: $\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ the projection maps) are not one-one, and hence **N** is pure-reducible.

Now, the homomorphic images of **N** are precisely those $\aleph_0$-semilattices which are isomorphic either to **N** or to a finite chain. Thus if $f:\mathbf{N}\to\Pi S_i$ ($i\in I$) is an embedding of **N** into a product of pure-irreducibles S_i such that, for each i, the composite of f with the ith projection map is *onto* S_i, then each S_i is a finite chain, and hence ΠS_i ($i\in I$) has a largest element, s, say. But then the homomorphism $\mathbf{N}[\{x\}]\to\Pi S_i$ over f which maps x to s contains $\{(n\wedge x, n)\mid n\in N\}$ in its kernel, whereas the existence of a homomorphism $\mathbf{N}[\{x\}]\to N$ over N containing $\{(n\wedge x, n)\mid n\in\mathbf{N}\}$ in its kernel would imply that **N** had a largest element. It follows that the Pure Representation Theorem (Taylor, [5, Theorem 3.6]) does not hold in $\boldsymbol{S}$.

2. **Injectivity.** Recall that $S\in\mathbf{S}$ is *injective* (in **S**) iff every homomorphism from a sub-$\aleph_0$-semilattice R of an $\aleph_0$-semilattice T into S has an extension to a homomorphism $T\to S$. Also, an extension T of an $\aleph_0$-semilattice S is *essential* iff every homomorphism $f:T\to R$ in **S** whose restriction to S is one-to-one is itself one-to-one, and T is an *injective hull* of S iff it is an essential, injective extension of S.

Bruns and Lakser [2] showed that a semilattice C is injective (in the class of all semilattices) iff it is a complete lattice and satisfies $a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$ for all $a \in A$ and $X \subseteq A$. A simple modification of their proofs yields the following characterization of injective $\aleph_0$-semilattices.

PROPOSITION 1. *An $\aleph_0$-semilattice S is injective (in* **S***) iff it is a complete lattice and satisfies*

(*) *for all countable families $(M_i)_{i\in\omega}$ of subsets of S,*

$$\bigwedge_{i\in\omega} \bigvee M_i = \bigvee_{\phi\in\Pi M_i} \bigwedge_{i\in\omega} \phi(i)$$

Proof. If S is injective in **S** then it is an absolute retract in **S**, and hence, by the remarks in §1, a retract of a power of **2**. Every power of **2** is a complete, completely distributive lattice and hence in particular satisfies (*); since **S**-homomorphisms preserve countable meets, it follows by an argument analogous to the proof of [2, Lemma 2] that every retract of a complete lattice satisfying (*) is also a complete lattice satisfying (*).

Conversely, if $S \in$ **S** satisfies the conditions of the Proposition, and if R is a sub-$\aleph_0$-semilattice of $T \in$ **S** and $f: R \to S$, then an argument analogous to the proof of [2, Lemma 1] shows that $g: T \to S$ given by $g(t) = \bigvee \{f(r) \mid r \leq t, r \in R\}$ is a homomorphism extending f.

COROLLARY. **S** *has enough injectives.*

In an equational class of finitary algebras which has enough injectives, every algebra which has no proper essential extensions is injective [1]. That this is not the case in the class of $\aleph_0$-semilattices is seen in the next proposition.

PROPOSITION 2. *For an $\aleph_0$-semilattice S, the following are equivalent;*

(1) *S has no proper essential extensions (in* **S***).*

(2) *The underlying semilattice of S is an injective semilattice.*

(3) *S is a retract of every extension which is singly generated over S.*

Proof. (1)⇒(2): For any $\aleph_0$-semilattice S, the canonical semilattice embedding of S into its semilattice-injective hull given in Bruns-Lakser [2] preserves arbitrary (existing) meets, and hence is an $\aleph_0$-embedding; since it is essential as a semilattice homomorphism it is essential as an $\aleph_0$-semilattice homomorphism. Thus if S has no proper essential extensions then it is equal to its semilattice-injective hull.

(2)⇒(3): Suppose that S satisfies (2), and that $T \supseteq S$ is an $\aleph_0$-semilattice extension generated over S by $\{a\}$. Then $T = S \cup \{u \wedge a \mid u \in S\}$. Let $X = \{s \in S \mid s \leq a\}$ and let $b = \bigvee_S X$, where $\bigvee_S$ denotes join in S. If, for $u, v \in S$, $u \wedge a = v \wedge a$ then for all $s \in X$, $u \wedge s = u \wedge s \wedge a = v \wedge s \wedge a = v \wedge s$, and consequently $u \wedge b = u \wedge \bigvee_S X = \bigvee_S \{u \wedge s \mid s \in X\} = \bigvee_S \{v \wedge s \mid s \in X\} = v \wedge \bigvee_S X = v \wedge b$. In particular, for $u \in S$, if $u \wedge a \in S$ then $u \wedge a = (u \wedge a) \wedge b = u \wedge b$. Thus we can define a function $f: T \to S$

by $f(u)=u$ for $u \in S$ and $f(u \wedge a)=u \wedge b$ for $u \in S$. It is easy to check that f preserves all at most countable meets, and hence is a retraction of T to S.

(3)$\Rightarrow$(1): Suppose S is a retract of each of its extensions which is singly generated over S. Let T be a proper extension of S, and let $R \subseteq T$ be a proper extension of S which is singly generated over S. Let V be an injective extension of S. The retraction of R onto S followed by the embedding $S \to V$ is a homomorphism $f: R \to V$ which is one-one on S but not one-one on R. By the injectivity of V, f has an extension $\bar{f}: T \to V$, and then $\bar{f}$ is one-one on S, and not one-one on T, which implies that T is not an essential extension of S. Thus S has no proper essential extensions.

COROLLARY. *Not every $\aleph_0$-semilattice has an injective hull* (*in* **S**).

Proof.. Any $\aleph_0$-semilattice which is not injective in **S**, but whose underlying semilattice is injective as a semilattice, has no injective hull in **S**. An example of such an $\aleph_0$-semilattice is the MacNeille completion M of the free Boolean algebra on countably many generators; every complete Boolean algebra is injective as a semilattice but it is easy to see that M does not satisfy the distributivity condition of Proposition 1.

3. **Equational Compactness.** An algebra A of characteristic m is *equationally compact* (see Nelson [4]) iff every subset $\Sigma \subseteq A[X]^2$ is contained in the kernel of a homomorphism $A[X] \to A$ over A whenever every subset of Σ with fewer than m elements already has this property (where $A[X]$ is the free extension of A by the set X in any equational class containing A). In particular, $S \in \mathbf{S}$ is equationally compact iff every subset $\Sigma \subseteq S[X]^2$ ($S[X]$ the free extension of S in **S** by the set X) is contained in the kernel of a homomorphism $S[X] \to S$ over S whenever every at most countable subset of Σ has this property.

Grätzer and Lakser showed that a semilattice A is equationally compact iff every non-empty subset of A has a meet, every up-directed subset of A has a join, and for every up-directed subset $X \subseteq A$ and all $a \in A$, $a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$. A modification of their techniques yields the following characterization of equationally compact $\aleph_0$-semilattices (where a partially ordered set is called $\aleph_1$-up-directed iff every at most countable subset has an upper bound):

PROPOSITION 3. *$S \in \mathbf{S}$ is equationally compact iff*

(1) *every non-empty subset of S has a meet in S*

(2) *every $\aleph_1$-up-directed subset of S has a join in S*

(3) *for all countable families $(M_i)_{i\in\omega}$ of $\aleph_1$-up-directed subsets of S,*

$$\bigwedge_{i\in\omega} \bigvee M_i = \bigvee_{\phi\in\Pi M_i} \bigwedge_{i\in\omega} \phi(i).$$

Proof. Suppose $S \in \mathbf{S}$ is equationally compact, and $T \subseteq S$ is a non-empty subset. Then every at most countable subset of

$$\Sigma = \{(t \wedge x, x) \mid t \in T\} \cup \{(u \wedge x, u) \mid u \leq t \quad \text{for all} \quad t \in T\} \subseteq S[\{x\}]^2$$

is contained in one of the form

$$\Sigma_{T'} = \{(t \wedge x, x) \mid t \in T'\} \cup \{(u \wedge x, u) \mid u \leq t \quad \text{for all} \quad t \in T\}$$

where T' is a non-empty, at most countable, subset of T. Since the homomorphism $S[\{x\}] \to S$ over S which maps x to $\bigwedge T'$ contains $\Sigma_{T'}$ in its kernel, it follows that there exists a homomorphism $f: S[\{x\}] \to S$ over S with $\Sigma \subseteq \operatorname{Ker} f$, and then $f(x) = \bigwedge T$.

A similar argument shows that every $\aleph_1$-up-directed subset of S has a join.

If $(M_i)_{i \in \omega}$ is a countable family of $\aleph_1$-up-directed subsets of S then $\{\bigwedge_{i \in \omega} \phi(i) \mid \phi \in \Pi M_i\}$ is $\aleph_1$-up-directed and hence has a join, a, say. Let

$$\Sigma = \bigcup_{i \in \omega} \{(x_i \wedge m, m) \mid m \in M_i\} \cup \{(a \wedge \bigwedge_{i \in \omega} x_i, \bigwedge_{i \in \omega} x_i)\} \subseteq S[X]^2$$

where $X = \{x_i \mid i \in \omega\}$ is a countable set disjoint from S.

If, for each i, M_i' is a countable subset of M_i, then in view of the $\aleph_1$-up-directedness of M_i there exists $m_i \in M_i$ with $m_i \geq m$ for each $m \in M_i'$. Also $\bigwedge_{i \in \omega} m_i \leq a$. Thus the homomorphism $f: S[X] \to S$ over S with $f(x_i) = m_i$ contains $\bigwedge_{i \in \omega} \{(x_i \wedge m, m) \mid m \in M_i'\} \cup \{(a \wedge \bigwedge_{i \in \omega} x_i, \bigwedge_{i \in \omega} x_i)\}$ in its kernel. Thus every at most countable subset of Σ is contained in the kernel of a homomorphism $S[X] \to S$ over S. By the equational compactness of S, there exists a homomorphism $h: S[X] \to S$ over S with $\Sigma \subseteq \operatorname{Ker} h$, and then $\bigwedge_{i \in \omega} \bigvee M_i \leq \bigwedge_{i \in \omega} h(x_i) \leq a$; the reverse inequality is trivial, and thus S satisfies (3).

For the converse, suppose $S \in \mathbf{S}$ satisfies (1)–(3) and that $\Sigma \subseteq S[X]^2$ such that every at most countable subset of Σ is contained in the kernel of a homomorphism $S[X] \to S$ over S. Then the $\aleph_0$-semilattice S^X also satisfies (1)–(3). For each $\phi \in S^X$, let $\bar{\phi}: S[X] \to S$ be the homomorphism over S extending ϕ. Then for any $\aleph_1$-up-directed subset $K \subseteq S^X$, the mapping $\phi_K: S[X] \to S$ given by $\phi_K(p) = \bigvee_{\phi \in K} \bar{\phi}(p)$ is a homomorphism (this follows from the distributivity condition (3)) over S and hence, since its restriction to X is $\bigvee K$, $\phi_K = \overline{\bigvee K}$. Consequently, for any $(p, q) \in S[X]^2$, if $K \subseteq S^X$ is an $\aleph_1$-up-directed subset such that $(p, q) \in \operatorname{Ker}(\bar{\phi})$ for each $\phi \in K$ then $(p, q) \in \operatorname{Ker}(\overline{\bigvee K})$.

Similarly if $K \subseteq S^X$ is a non-empty subset with $(p,q) \in \operatorname{Ker}(\bar{\phi})$ for each $\phi \in K$ then $(p, q) \in \operatorname{Ker}(\overline{\bigwedge K})$.

Now, by assumption, for each at most countable $T \subseteq \Sigma$, $K_T = \{\phi \in S^X \mid T \subseteq \operatorname{Ker}(\bar{\phi})\} \neq \varnothing$. It follows from the above remark that $T \subseteq \operatorname{Ker}(\overline{\bigwedge K_T})$. Let $\phi_T = \bigwedge K_T$. Then, for each $(p, q) \in \Sigma$, $\{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\}$ is an $\aleph_1$-up-directed subset of S^X and $(p, q) \in \operatorname{Ker}(\bar{\phi}_T)$ for each at most countable $T \subseteq \Sigma$ with $(p, q) \in T$, and hence, by the above remarks, $(p, q) \in \operatorname{Ker}(\bar{\phi})$ where $\phi = \bigvee \{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\}$. However, for each $(p, q) \in \Sigma$,

$$\bigvee \{\phi_T \mid (p, q) \in T \subseteq \Sigma, |T| \leq \aleph_0\} = \bigvee \{\phi_T \mid T \subseteq \Sigma, |T| \leq \aleph_0\}$$

(this follows essentially from the fact that if T is at most countable so is $T\cup\{(p,q)\}$) and hence $\Sigma\subseteq\operatorname{Ker}\psi$ where $\psi:S[X]\to S$ is the homomorphism over S whose restriction to X is $\bigvee\{\phi_T \mid T\subseteq\Sigma,\ |T|\leq\aleph_0\}$. Thus S is equationally compact.

NOTE. The above methods can be used to show that $S\in\mathbf{S}$ is (in the terminology of Grätzer-Lakser [3]), 1-equationally compact (i.e. every subset $\Sigma\subseteq S[\{x\}]^2$ is contained in the kernel of a homomorphism $S[\{x\}]\to S$ over S whenever every at most countable subset has this property) iff it satisfies (1) and (2) of Proposition 2, and $s\wedge\bigvee T=\bigvee\{s\wedge t \mid t\in T\}$ for all $s\in S$, all $\aleph_1$-up-directed $T\subseteq S$. Thus, in contrast to the situation for semilattices, 1-equationally compact is not equivalent to equationally compact in $\mathbf{S}$.

COROLLARY 1. *Every at most countable $\aleph_0$-semilattice is equationally compact.*

COROLLARY 2. *Every $\aleph_0$-semilattice of at most countable height in which every non-empty subset has a meet is equationally compact.*

Note added in proof: All of the foregoing works just as well with $\aleph_0$ replaced by *any* infinite cardinal $\mathscr{M}$, giving characterizations of the injective and the equationally compact $\mathscr{M}$-semilattices. By combining these results, or by analagous arguments, one concludes that the injectives in the category of all complete semilattices are precisely the complete, completely distributive semilattices, and hence (since the latter condition is self-dual) these also coincide with the projective complete semilattices. These results on *complete* semilattices were presented by A. Waterman in a seminar at McMaster in 1966, and also appeared in Crown [Math. Annalen. **187** (1970) 295–299].

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ON SECOND-ORDER DIFFERENTIAL OPERATORS WITH BOHR-NEUGEBAUER TYPE PROPERTY

BY

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ABSTRACT. Let B be a bounded linear operator having domain and range in a Banach space. If the second-order differential operator d^2/dt^2-B has a Bohr-Neugebauer type property for Bochner almost periodic functions, then any Stepanov-bounded solution of the differential equation $(d^2/dt^2)u(t)-Bu(t)=g(t)$ is Bochner almost periodic, with $g(t)$ being a Stepanov-almost periodic continuous function.

1. Suppose X is a Banach space and J is the interval $-\infty<t<\infty$. A function $f\in L^p_{\mathrm{loc}}(J;X)$ with $1\le p<\infty$ is said to be Stepanov-bounded or S^p-bounded on J if

$$\|f\|_{S^p}=\sup_{t\in J}\left[\int_t^{t+1}\|f(s)\|^p\,ds\right]^{1/p}<\infty \tag{1.1}$$

(for the definitions of (Bochner or strong) almost periodicity and S^p-almost periodicity, see pp. 3 and 77, Amerio-Prouse [1]).

Suppose that B is a bounded linear operator having domain and range in X. We say that the second-order differential operator d^2/dt^2-B has Bohr-Neugebauer property if, for any almost periodic X-valued function $f(t)$, any bounded (on J) solution of the equation

$$\frac{d^2}{dt^2}u(t)-Bu(t)=f(t)\quad\text{on}\quad J \tag{1.2}$$

is almost periodic.

The object of this paper is to establish the following result.

THEOREM. *For a bounded linear operator B with domain $D(B)$ and range $R(B)$ in a Banach space X, let the differential operator d^2/dt^2-B be such that, for any almost periodic X-valued function $f(t)$, any S^p-bounded solution $u:J\to D(B)$ of the equation* (1.2) *is S^1-almost periodic. If $p>1$, then, for any S^1-almost periodic continuous X-valued function $g(t)$, any S^p-bounded solution $u:J\to D(B)$ of the equation*

$$\frac{d^2}{dt^2}u(t)-Bu(t)=g(t)\quad\text{on}\quad J \tag{1.3}$$

is almost periodic.

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2. We shall require the following result.

Lemma (A). *If a differentiable function $h: J \to X$ is S^1-almost periodic, and if h' is uniformly continuous on J, then h and h' are both almost periodic from J to X.*

Proof. See Remark (iii) of Rao-Hengartner [5].

3. **Proof of Theorem.** By (1.3), we have the representation

$$u'(t) = u'(0) + \int_0^t Bu(s)\,ds + \int_0^t g(s)\,ds \quad \text{on} \quad J. \tag{3.1}$$

From this representation, we can show that $u'(t)$ is uniformly continuous on J (see Rao [4]).

Now consider a sequence $\{\rho_n(t)\}_{n=1}^{\infty}$ of non-negative continuous functions on J such that

$$\rho_n(t) = 0 \quad \text{for} \quad |t| \geq n^{-1}, \int_{-n^{-1}}^{n^{-1}} \rho_n(t)\,dt = 1. \tag{3.2}$$

The convolution between u and ρ_n is defined by

$$(u * \rho_n)(t) = \int_J u(t-s)\rho_n(s)\,ds = \int_J u(s)\rho_n(t-s)\,ds. \tag{3.3}$$

From (1.3), it follows that

$$\frac{d^2}{dt^2}(u * \rho_n)(t) - B(u * \rho_n)(t) = (g * \rho_n)(t) \quad \text{on} \quad J. \tag{3.4}$$

As shown in Rao [4], $(u * \rho_n)(t)$ is bounded on J for all $n \geq 1$ and $(g * \rho_n)(t)$ is almost periodic from J to X for all $n \geq 1$.

Therefore it follows from our assumption on the operator $d^2/dt^2 - B$ that $(u * \rho_n)(t)$ is S^1-almost periodic from J to X for all $n \geq 1$. The only use made of this assumption in this paper is to guarantee that $(u * \rho_n)(t)$ is almost periodic in the Stepanov sense.

Further, by the uniform continuity of $u'(t)$ on J, we can show that $(u' * \rho_n)(t)$ is uniformly continuous on J.

Now we have

$$(u * \rho_n)'(t) = (u' * \rho_n)(t) \quad \text{on} \quad J. \tag{3.5}$$

Hence $(u * \rho_n)'(t)$ is uniformly continuous on J. Thus, by Lemma (A), $(u * \rho_n)(t)$ and $(u' * \rho_n)(t)$ are both almost periodic from J to X for all $n \geq 1$.

Again by the uniform continuity of $u'(t)$ on J, the sequence of convolutions $(u' * \rho_n)(t)$ converges to $u'(t)$ as $n \to \infty$, uniformly on J. So $u'(t)$ is almost periodic from J to X, and hence is bounded on J. Therefore $u(t)$ is uniformly continuous on J. Consequently, $(u * \rho_n)(t) \to u(t)$ as $n \to \infty$, uniformly on J. So $u(t)$ is almost periodic from J to X, which completes the proof of the theorem.

4. **Notes.** (i) For $p=1$, our Theorem remains valid for any S^1-bounded uniformly continuous solution of the equation (1.3).

(ii) Suppose X is a separable Hilbert space, and consider the second-order operator differential equation

$$\frac{d^2}{dt^2}u(t)-Bu(t)=f(t) \quad \text{on} \quad J, \quad \text{where} \quad f\colon J\to X \quad \text{is}$$

an almost periodic function and B is a completely continuous linear operator in X commuting with its adjoint (see p. 258, Bochner-Neumann [2]). Then, by Theorem 1 of Cooke [3], the operator d^2/dt^2-B has Bohr-Neugebauer property. Now suppose that $u(t)$ is an S^p-bounded solution ($1<p<\infty$) of the above differential equation.

If we replace g by f in the proof of our Theorem, then, by the Bohr-Neugebauer property of the operator d^2/dt^2-B, it follows that $u(t)$ is almost periodic from J to X. Thus, in this case, the operator d^2/dt^2-B satisfies the assumption of our Theorem for $p>1$.

(iii) Now suppose X is a Hilbert space and B is a bounded linear operator in X with $B\geq 0$. Then the operator d^2/dt^2-B has Bohr-Neugebauer property (see Zaidman [6]). Consequently, the operator d^2/dt^2-B satisfies the hypothesis of our Theorem for $p>1$.

(iv) Finally, suppose X is a reflexive Banach space and $B=0$. Given an almost periodic X-valued function $f(t)$, suppose $u(t)$ is a bounded solution of the differential equation

$$\frac{d^2}{dt^2}u(t)=f(t) \quad \text{on} \quad J. \tag{4.1}$$

Then we have the representation

$$u'(t)=u'(0)+\int_0^t f(s)\,ds \quad \text{on} \quad J. \tag{4.2}$$

By Lemma 2 of Cooke [3], it follows from (4.1) that $u'(t)$ is bounded on J. Consequently, by (4.2), $u'(t)$ is almost periodic from J to X (see Amerio-Prouse [1], p. 55 and Authors' Remark on p. 82). Therefore $u(t)$ is also almost periodic from J to X. Hence the operator d^2/dt^2 has Bohr-Neugebauer property.

Now, given an S^1-almost periodic continuous X-valued function $g(t)$, suppose $u(t)$ is an S^p-bounded solution ($1\leq p<\infty$) of the differential equation

$$\frac{d^2}{dt^2}u(t)=g(t) \quad \text{on} \quad J. \tag{4.3}$$

From (4.3), it follows that

$$\frac{d^2}{dt^2}(u*\rho_n)(t)=(g*\rho_n)(t) \quad \text{on} \quad J, \tag{4.4}$$

where $\{\rho_n(t)\}_{n=1}^{\infty}$ is the sequence defined in the proof of our Theorem.

Then $(u * \rho_n)(t)$ is bounded on J and $(g * \rho_n)(t)$ is almost periodic from J to X. As shown above, $(u * \rho_n)(t)$ and $(u * \rho_n)'(t) = (u' * \rho_n)(t)$ are both almost periodic from J to X.

By (4.3), it follows from Theorem 8, p. 79, Amerio-Prouse [1] that $u'(t)$ is uniformly continuous on J. So $(u' * \rho_n)(t) \to u'(t)$ as $n \to \infty$, uniformly on J. Hence $u'(t)$ is almost periodic from J to X. Therefore $u(t)$ is uniformly continuous on J, and hence $(u * \rho_n)(t) \to u(t)$ as $n \to \infty$, uniformly on J. Consequently, $u(t)$ is almost periodic from J to X. So the operator d^2/dt^2 satisfies the assumption of our Theorem for $1 \leq p < \infty$.

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AN APPLICATION OF PAIRWISE INDEPENDENCE OF RANDOM VARIABLES TO REGRESSION ANALYSIS

BY

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ABSTRACT. The exact probability of making an error at the first stage of a stepwise regression is found, both when the independent variables are fixed and when they are random. The probabilities differ in the two cases because the random variables involved are pairwise independent but not jointly independent.

1. **Introduction.** In linear regression, an F-statistic (or t-statistic) is used to test the hypothesis that the slope is zero. The null distribution of each of these statistics does not depend on whether the independent variable is random or is fixed by the experimenter. The purpose of this paper is to show that the property does not carry over to multilinear regression and that the explanation for this lies in the pairwise independence but not joint independence of the various random variables that are involved. As well, it may serve to bring more attention to a "natural" example of random variables that are pairwise independent but not jointly independent, an example that does not seem to have achieved much notice.

Lastly, a 1971 result of Draper, Guttman and Kanemasu [3] is generalized to include non-orthogonal estimators in the case where the independent variables are fixed.

2. **Preliminaries.** Throughout this paper, we will use the following convention. Capital letters will represent random variables. Small letters will represent fixed variables or values of random variables. For example, if X is a random variable, $E(Y \mid x)$ will represent the conditional expected value of the random variable Y, given $X=x$. Otherwise, $E(Y \mid x)$ will imply that the expected value of Y depends functionally on a (non-random) variable x.

If $Y, X_1, \ldots, X_k$ are random variables, we define $\mu_0=EY, \mu_i=EX_i (i=1, \ldots, k)$, $\sigma_{00}=VY, \sigma_{ii}=VX_i\ (i=1, \ldots, k), \sigma_{0i}=\text{Cov}(Y, X_i)\ (i=1, \ldots, k), \sigma_{ir}=\text{Cov}(X_i, X_r)$ $(i, r=1, \ldots, k; i\neq r)$. We let $\boldsymbol{\mu}=(\mu_0, \ldots, \mu_k)$ and $\underset{\sim}{\Sigma}$ be the $k+1$ by $k+1$ matrix whose (i,j)th entry is $\sigma_{ij} (i, j=0, \ldots, k)$. We let Σ_{ij} be the cofactor of σ_{ij} in $\underset{\sim}{\Sigma}$ and let Σ be determinant of $\underset{\sim}{\Sigma}$.

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Assume the following model. Y is a normal random variable such that

$$E(Y \mid \mathbf{x}) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i \tag{2.1}$$

$$V(Y \mid \mathbf{x}) = \sigma_{0|1\cdots k} \tag{2.2}$$

where $\mathbf{x}=(x_1, \ldots, x_k)$. If $(Y, \mathbf{X})$ has a multivariate normal distribution with parameters $\boldsymbol{\mu}$ and $\underset{\sim}{\Sigma}$ ($\underset{\sim}{\Sigma}$ positive definite), then

$$\beta_0 = \mu_0 - \sum_{i=1}^{k} \beta_i \mu_i, \tag{2.3}$$

$$\beta_i = [(\sigma_{01}, \ldots, \sigma_{0k})\underset{\sim}{\Sigma}_0^{-1}]_i \qquad i = 1, \ldots, k, \tag{2.4}$$

$$\sigma_{0|1,\ldots,k} = \sigma_{00} - (\sigma_{01}, \ldots, \sigma_{0k})\underset{\sim}{\Sigma}_0^{-1}(\sigma_{01}, \ldots, \sigma_{0k})^T \tag{2.5}$$

where $\underset{\sim}{\Sigma}_0$ is obtained from $\underset{\sim}{\Sigma}$ by deleting its first row and first column. In any case, we assume that $-\infty<\beta_i<+\infty$ ($i=0, 1, \ldots, k$) and that $\sigma_{0|1,\ldots,k}>0$. Within this model, let

$$\Omega = \{\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k): -\infty < \beta_i < +\infty, \qquad i = 1, \ldots, k\},$$

let $\omega_i^1=\{\boldsymbol{\beta} \in \Omega: \beta_i=0\}$ and $\omega_i^{-1}=\{\boldsymbol{\beta} \in \Omega: \beta_i\neq 0\}$ ($i=1, 2, \ldots, k$) and consider the 2^k subsets of Ω given by $\bigcap_{i=1}^{k} \omega_i^{x_i}$ as $x_i=\pm 1$, $i=1, 2, \ldots, k$. These 2^k subsets form a partition of Ω from which we wish to choose one subset and assert that β lies in it, i.e. we wish to assert some subset of the β's are zero and the rest non-zero. This choice will be based on independent observations $Y_1, Y_2, \ldots, Y_n$ associated with vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ ($n>k$) where $\mathbf{x}_i$ is the value of $\mathbf{X}_i$ if the vector $(Y_i, \mathbf{X}_i)$ is multivariate normal ($i=1, 2, \ldots, n$).

In Chapter 6 of Draper and Smith [4], six possible methods are listed for choosing one of the subsets of Ω. In this paper, we discuss some problems that arise from the forward selection procedure, the stepwise procedure, and variations thereof. These procedures successively add one more β at each stage to those that are stated as being non-zero until a stage is reached at which any remaining β's are stated to be zero. At stage one, either one β is asserted to be non-zero or all β's are asserted to be zero.

To describe the mechanics of stage one, we assume, for the moment, that $(Y, \mathbf{X})$ is multivariate normal with $\underset{\sim}{\Sigma}=\underset{\sim}{D}$ where $\underset{\sim}{D}$ is a diagonal matrix and that $(Y_1, \mathbf{X}_1), (Y_2, \mathbf{X}_2), \ldots, (Y_n, \mathbf{X}_n)$ are a random sample on $(Y, \mathbf{X})$. We let

$$S_{00} = \sum_{j=1}^{n}(Y_j - \bar{Y})^2, \qquad S_{ii} = \sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2, \qquad S_{0i} = \sum_{j=1}^{n}(Y_j - \bar{Y})(X_{ij} - \bar{X}_i)$$

and

$$S_{ir} = \sum_{j=1}^{n}(X_{ij} - \bar{X}_i)(X_{rj} - \bar{X}_r)$$

where $i, r=1, \ldots, k$ $(i \neq r)$. Now let

$$R_i = S_{0i}/(S_{ii}^{1/2} S_{00}^{1/2}) \tag{2.6}$$

$$F_i = \frac{(n-2)S_{0i}^2/S_{ii}}{S_{00}-S_{0i}^2/S_{ii}} \tag{2.7}$$

for $i=1, \ldots, k$. It is easily seen that $F_i=(n-2)R_i^2/(1-R_i^2)$ so that F_i and R_i^2 are 1-1 monotone increasing functions of each other. At stage one, one looks at $S_{01}^2/S_{11}, \ldots, S_{0k}^2/S_{kk}$ and chooses the largest quantity, say S_{0i}^2/S_{ii}. It is conventional to assert that $\beta_i \neq 0$ if and only if, for the corresponding F_i, $F_i \geq f_{1,n-2,\alpha}$ where $f_{1,n-2,\alpha}$ is the upper 100α percent point for the central $F(1, n-2)$ distribution. Otherwise, all the β's are said to be zero.

If we let $S^*=\max(S_{01}^2/S_{11}, \ldots, S_{0k}^2/S_{kk})$ and let $F^*=(n-2)S^*/(S_{00}-S^*)$, then

$$P(F^* \geq f_{1,n-2,\alpha} \mid \underset{\sim}{\Sigma} = \underset{\sim}{D}) \tag{2.8}$$

is the probability of making the error at stage one of saying that some β is non-zero, when, in fact, all the β's are zero.

By inspection, it is seen that S_{0i}^2/S_{ii} is largest if and only if R_i^2 is largest (amongst the R's) if and only if F_i^2 is largest (amongst the F's). Thus,

$$\{F^* < f_{1,n-2,\alpha}\} = \bigcap_{i=1}^{k} \{F_i < f_{1,n-2,\alpha}\}. \tag{2.9}$$

In later sections, we show that the F_i are jointly independent central $F(1, n-2$ random variables when $\underset{\sim}{\Sigma}=\underset{\sim}{D}$. Thus,

$$\begin{aligned} P(F^* \geq f_{1,n-2,\alpha} \mid \underset{\sim}{\Sigma} = \underset{\sim}{D}) &= 1-[P(F < f_{1,n-2,\alpha})]^k \\ &= 1-(1-\alpha)^k \\ &\cong k\alpha. \end{aligned} \tag{2.10}$$

The procedure for choosing a non-zero β in the fixed case is the same but we show that the probability corresponding to (2.8) is different and is a function of the fixed x's, as well as n and α.

3. Bilinear regression. Since the formulas are easiest when $k=2$, we study this problem in more detail. Recall that $n>2$. Let H_0 be $\omega_1^1 \cap \omega_2^1$, i.e., $H_0:\beta_1=\beta_2=0$.

THEOREM 3.1. *If* (Y, X_1, X_2) *is a multivariate* $N(\mu, \underset{\sim}{\Sigma})$ *random vector with* $\underset{\sim}{\Sigma}$ *positive definite and* (Y_1, Y_{11}, X_{21}), $(Y_2, X_{12}, X_{22}), \ldots, (Y_n, X_{1n}, X_{2n})$ *are a random sample on* (Y, X_1, X_2), *then* R_1 *and* R_2 *are independent under* H_0, *with joint density*

$$f(r_1, r_2) = f(r_1)f(r_2) \tag{3.1}$$

on $-1<r_1, r_2<+1$ where $f(r_i)=\Gamma((n-1)/2)(1-r_i^2)^{(n-4)/2}/\Gamma(\frac{1}{2})\Gamma((n-2)/2)$, $i=1, 2$. Likewise F_1, F_2 are independent $F(1, n-2)$ random variables under H_0.

Proof. Using (2.4), we can write

$$\beta_1 = \frac{\sigma_{01}\sigma_{22}-\sigma_{02}\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \tag{3.2}$$

$$\beta_2 = \frac{\sigma_{02}\sigma_{11}-\sigma_{01}\sigma_{12}}{\sigma_{01}\sigma_{22}-\sigma_{12}^2} \tag{3.3}$$

where the denominators are positive since $\underset{\sim}{\Sigma}$ is positive definite. Under H_0, the numerators are zero. If $\sigma_{01}\neq 0$, $\sigma_{02}\neq 0$, and $\sigma_{12}\neq 0$, then $\sigma_{12}^2=\sigma_{11}\sigma_{22}$, as can be quickly computed. Since this is impossible, $\sigma_{12}=0$ and this implies $\sigma_{01}=\sigma_{02}=0$. Thus, from (29.13.1) on p. 411 of Cramer [2], the joint density of R_1, R_2 and $R=S_{12}/(S_{11}^{1/2}S_{22}^{1/2})$, under H_0 is

$$f(r_1, r_2, r) = \frac{\Gamma\left(\frac{n-1}{2}\right)^2}{\Gamma(\frac{1}{2})^3\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-3}{2}\right)}\ (1-r_1^2-r_2^2-r^2+2r_1r_2r)^{(n-5)/2} \tag{3.4}$$

on $-1<r_1, r_2, r<1$ and $1-r_1^2-r_2^2-r^2-2r_1r_2r>0$. Integrating out r [using the change of variable $u=(r-r_1r_2)/(1-r_1^2)^{1/2}(1-r_2^2)^{1/2}$] yields (3.1). Thus, R_1 and R_2 are independent under H_0. Hence, F_1 and F_2 are independent $F(1, n-2)$ random variables under H_0 because of the functional relationship between F and R.

From Theorem 3.1, we see that (2.8) can be written as

$$\begin{aligned} P_{H_0}(F^*\geq f_{1,n-2,\alpha}) &= P_{H_0}(F_1\geq f_{1,n-2,\alpha} \quad\text{or}\quad F_2\geq f_{1,n-2,\alpha}) \\ &= 1-(1-\alpha)^2. \end{aligned} \tag{3.5}$$

Formula (3.5) does not follow when the **x**'s are fixed as the following theorem shows. Let $Y_1, \ldots, Y_n$ be independent normal random variables associated with vectors $(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})$. With some abuse of notation, let $S_{00}=\sum_{i=1}^n (Y_i-\overline{Y})^2$, $S_{0i}=\sum_{j=1}^n (Y_j-\overline{Y})(x_{ij}-\bar{x}_i)$, $i=1, 2$, $s_{ii}=\sum_{j=1}^n (x_{ij}-\bar{x}_i)^2$, $i=1, 2$, and $s_{12}=\sum_{j=1}^n (x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2)$. Let $R_i=S_{0i}/S_{00}^{1/2}s_{ii}^{1/2}$, $i=1, 2$, $r=s_{12}/s_{11}^{1/2}s_{22}^{1/2}$, and $F_i=(n-2)R_i^2/(1-R_i^2)$, $i=1, 2$. With this notation, we have the following theorem.

THEOREM 3.2. *If* $(1, 1, \ldots, 1)$, $\mathbf{x}_1=(x_{11}, x_{12}, \ldots, x_{1n})$, *and* $\mathbf{x}_2=(x_{21}, x_{22}, \ldots, x_{2n})$ *are fixed (linearly independent) n-vectors such that* $Y_1, Y_2, \ldots, Y_n$ *are independent normal random variables with*

$$\left.\begin{aligned} &E(Y_j \mid \mathbf{x}_1, \mathbf{x}_2) = \beta_0+\beta_1x_{1j}+\beta_2x_{2j} && (3.6)\\ &V(Y_j \mid \mathbf{x}_1, \mathbf{x}_2) = \sigma_1^2, && (3.7)\end{aligned}\right\} \quad j = 1, 2, \ldots, n$$

then R_1, R_2, *and* S_{00} *have joint density*

$$f(r_1, r_2, s \mid \mathbf{x}_1, \mathbf{x}_2) = \frac{(1-r_1^2-r_2^2-r^2+2r_1r_2r)^{(n-5)/2}s^{(n-3)/2}e^{-(Q)/}2\sigma_1^2}{(2\sigma_1^2)^{(n-1)/2}\Gamma(\frac{1}{2})^2\Gamma\left(\frac{n-3}{2}\right)(1-r^2)^{(n-4)/2}} \tag{3.8}$$

where $Q=s-2\beta_1 s_{11}^{1/2} r_1^{1/2} s^{1/2}-2\beta_2 s_{22}^{1/2} r_2^{1/2} s^{1/2}+2\beta_1\beta_2 s_{12}+\beta_1^2 s_{11}+\beta_2^2 s_{22}$ *on* $s>0$, $-1<r_1, r_2<1$, $1-r_1^2-r_2^2-r^2+2r_1r_2r>0$.

In particular, under H_0, R_1 *and* R_2 *(hence* F_1 *and* F_2*) are dependent random variables and* R_1 *and* R_2 *have joint density*

$$f(r_1, r_2 \mid \mathbf{x}_1, \mathbf{x}_2) = \frac{\Gamma\left(\frac{n-1}{2}\right)(1-r_1^2-r_2^2-r^2+2r_1r_2r)^{(n-5)/2}}{\Gamma(\frac{1}{2})^2\Gamma\left(\frac{n-3}{2}\right)(1-r^2)^{(n-4)/2}} \tag{3.9}$$

on $-1<r_1, r_2<1$ *and* $1-r_1^2-r_2^2-r^2+2r_1r_2r>0$.

Proof. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are fixed, the joint density of F_1 and F_2 can be found directly by an extension of the matrix method that shows the independence of $\bar{X}$ and s^2 when $X_1, X_2, \ldots, X_n$ are a random sample on X, a normal random variable. Let A be the orthonormal matrix formed from $(1, 1, \ldots, 1)$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $n-3$ additional linearly independent n-vectors. (Recall that $(1, 1, \ldots, 1)$, $\mathbf{x}_1$, and $\mathbf{x}_2$ are linearly independent vectors.) Using the fact that $\mathbf{U}=\underset{\sim}{A}\mathbf{Z}$ is $N(\mathbf{O}, I)$ if $\mathbf{Z}$ is $(\mathbf{O}, I)$, changing variables carefully, and integrating out extraneous variables yields the joint density of R_1, R_2, and S_{00} as

$$f(r_1, r_2, s \mid \mathbf{x}_1, \mathbf{x}_2) = \frac{(1-r_1^2-r_2^2-r^2+2r_1r_2r)^{(n-5)/2} s^{(n-3)/2} e^{-Q/2\sigma_1^2}}{(2\sigma_1^2)^{(n-1)/2}\Gamma(\frac{1}{2})^2\Gamma\left(\frac{n-3}{2}\right)(1-r^2)^{(n-4)/2}} \tag{3.10}$$

where $Q=s-2\beta_1 s_{11}^{1/2} r_1^{1/2} s^{1/2}-2\beta_2 s_{22}^{1/2} r_2^{1/2} s^{1/2}+2\beta_1\beta_2 s_{12}+\beta_1^2 s_{11}+\beta_2^2 s_{22}$ on the appropriate ranges. Note that this density depends on $\mathbf{x}_1$ and $\mathbf{x}_2$ only through s_{11}, s_{22}, and s_{12}.

Equation (3.8) can be obtained in an alternate way by assuming that the $\mathbf{x}$'s, though fixed, are values of corresponding normal random variables. That is, write out the Wishart distribution for S_{00}, S_{01}, S_{02}, S_{11}, S_{12}, and S_{22} in terms of variances and covariances, divide it by the Wishart distribution of S_{11}, S_{12}, and S_{22}, and obtain the distribution of S_{00}, S_{01}, S_{02}, given $(S_{11}, S_{12}, S_{22})=(s_{11}, s_{12}, s_{22})$. Use (2.4) and (2.5) to change the constants from variances and covariances to β's and $\sigma_1^2=\sigma_{0|1,\ldots,k}^2$. Finally, use the mapping $(S_{00}, S_{01}, S_{02})\rightarrow(S_{00}, R_1, R_2)$ to show that the conditional density of S_{00}, R_1, and R_2 is (3.8).

If H_0 is true, inspection of (3.10) shows that S_{00} is independent of R_1 and R_2, the joint density of latter depends on $\mathbf{x}_1$ and $\mathbf{x}_2$ only through r, their sample correlation coefficient, and the density is given by

$$f(r_1, r_2 \mid \mathbf{x}_1, \mathbf{x}_2) = \frac{\Gamma\left(\frac{n-1}{2}\right)(1-r_1^2-r_2^2-r^2+2r_1r_2r)^{(n-5)/2}}{\Gamma(\frac{1}{2})^2\Gamma\left(\frac{n-3}{2}\right)(1-r^2)^{(n-4)/2}} \tag{3.11}$$

on the appropriate range. Thus, under H_0, R_1 and R_2 (hence F_1 and F_2) are dependent and the proof is complete.

The reason for this basic difference between the random and fixed cases can be seen by looking at equations (3.1) and (3.4). From these, it follows that the random variables R_1, R_2, and R are pairwise independent but not jointly independent as was first shown by Geisser and Mantel [6] in 1962. When the independent variables are random, R_1 and R_2 (hence F_1 and F_2) are independent, under H_0. When the independent variables are fixed, r is known and the conditional distribution of R_1 and R_2, obtainable by dividing (3.4) by the density of R, is not factorable, even under H_0.

Most elementary texts in probability and statistics stress the fact that pairwise independence does not, in general, imply total independence. But natural examples are hard to find, particularly examples involving continuous random variables. (See Feller [5] p. 116–117.) Thus, Geisser and Mantel's example is of valuable pedagogical use and, in the present setup, explains why different results should be expected in the fixed and random cases.

For $k=2$, we have found P_{H_0} (H_0 is rejected) both in the random and in the fixed cases. In the former case, the error probability can be set at α by setting the critical level at $f_{1,n-2,1-\sqrt{1-\alpha}}$. In the latter case, for any critical level c, we have

$$\begin{aligned} P_{H_0}\,(H_0 \text{ is rejected}) &= P_{H_0}(F_1 \geq c \quad \text{or} \quad F_2 \geq c) \\ &= P_{H_0}(R_1^2 \geq c' \quad \text{or} \quad R_2^2 \geq c') \\ &= 1-\int_{-c'}^{c'}\int_{-c'}^{c'} f(r_1, r_2 \mid \mathbf{x}_1, \mathbf{x}_2)\, dr_1\, dr_2 \end{aligned} \tag{3.12}$$

where $c'=c/(n-2+c)$ and this probability depends on c, n, and r (the correlation coefficient of $\mathbf{x}_1$ and $\mathbf{x}_2$). Note that $r^2<1$ by our assumption that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

In 1971, Draper *et al.* [3] found the probability given in (3.12) for a special case. They assume that the estimators of the β's are orthogonal. Inspection of the covariance matrix shows that this implies that r, the sample correlation coefficient of the x_1's and x_2's, is zero. Orthogonality actually implies more but (3.12) depends on the x's only through r. Thus, for various n's and α's, they find that the error probabilities behave approximately as they would if the F's were independent when $r=0$.

If the independent variables are fixed by the experimenter, the probabilities and/or critical levels must be found by numerical means but it would seem wiser to compute these quantities as needed rather than preparing tables since these would be simply too large. However, computations to see how the probability in (3.12) depends on the value of r would be of interest.

4. Multilinear regression. For $k>2$ and $n>k$, similar results hold but the equations become unwieldy. If $(Y, X_1, \ldots, X_k)$ is a multivariate $N(\boldsymbol{\mu}, \underset{\sim}{\Sigma})$ random vector with $\underset{\sim}{\Sigma}=\underset{\sim}{D}$, a diagonal matrix that is positive definite, then $R_1, R_2, \ldots, R_k$ are jointly independent. (See, for example, Anderson [1], p. 177, Problem 9). Thus, the problem of finding $P_{H_0}(F_1 \geq c$ or $F_2 \geq c$ or $\cdots$ or $F_k \geq c)$ for fixed c and the problem of finding c so as to fix the above probability are easily solved. There are, however $k(k+1)/2$ correlation coefficients and they are jointly dependent. Since Y holds no special place in this vector, it follows that amongst the $k(k+1)/2$ jointly dependent random variables, there are $k+1$ sets of k random variables that are jointly independent, each correlation coefficient appearing exactly twice in the $k+1$ sets. This implies that the $k(k+1)/2$ correlation coefficients are pairwise independent, as pointed out by Geisser and Mantel [6]. More importantly, it shows that k-independence of certain subsets of $k(k+1)/2$ random variables is still not enough to ensure the joint independence of the full set.

Thus, in the general case where the dependent variables are random, if H_0 is $\bigcap_{i=1}^{k} \omega_1^i$, i.e., $H_0: \beta_1 = \cdots = \beta_k = 0$, then (2.8) can be written as

$$\begin{aligned} P_{H_0}(H_0 \text{ is rejected}) &= P_{H_0}(F^* \geq f_{1,n-2,\alpha}) \\ &= 1 - P_{H_0}(F_1 < f_{1,n-2,\alpha}, \ldots, F_k < f_{1,n-2,\alpha}) \\ &= 1-(1-\alpha)^k \\ &\cong k\alpha, \end{aligned} \tag{4.1}$$

for α small.

If the independent variables are fixed by the experimenter, the easiest way to obtain the conditional distribution of $R_1, R_2, \ldots, R_m$, under H_0, is to divide the joint density of all $k(k+1)/2$ correlation coefficients by the joint density of the $(k-1)k/2$ correlation coefficients that do not involve Y. If $\underset{\sim}{R}=(r_{ij})$ is a $k+1$ by $k+1$ matrix $(i, j=0, 1, \ldots, k)$ with determinant $R=|\underset{\sim}{R}|$ and R_{00} is the cofactor of r_{00}, the conditional density of $R_1, R_2, \ldots, R_k$ is given by

$$f(r_{01}, \ldots, r_{0k} \mid r_{12}, \ldots, r_{k-1,k}) = \frac{C(k+1, n)}{C(k, n) R_{00}^{1/2}} \left(\frac{R}{R_{00}}\right)^{(n-k-3)/2} \tag{4.2}$$

on the set where R is positive, the density being defined whenever R_{00} is positive, and $C(k+1, n)/C(k, n)$ being a constant so that the integral of (8) over $r_{01}, \ldots, r_{0k}$ is one.

For a fixed significance level c, it is now theoretically possible to compute the probability of rejecting H_0, given H_0 is true. Likewise, it is theoretically possible to determine c so that the stated probability is α. But both problems can only be solved by numerical methods, preferably on a computer. In [3], it is shown that the R's behave as if they are approximately independent when the estimators are orthogonal.

5. **Problems.** The results of this paper leave many problems. Amongst them are the following:

(1) Is the error probability relatively uninfluenced by the choice of r when the

independent variables are fixed ($k=2$), i.e., are the F's approximately independent no matter what the choice of the x's.

(2) For fixed n and $\alpha(k=2)$, can r be chosen, in the fixed case, so as to minimize the error probability? Are there values of r for which the error probability is less than the probability when the independent variables are random?

(3) More generally, this paper only investigates the probability of one error for one test. The other errors are functions of the non-zero β's. Does this test perform well, at least as $|\boldsymbol{\beta}|$ becomes large? The procedures mentioned in the introduction use at least k tests in stage one. Can a procedure be devised that uses only k tests in total and, for fixed probability of rejecting H_0 when H_0 is true, improves the probability of finding other hypotheses when they are true?

(4) In some applied areas, e.g. economics, the independent variables are usually considered random and k is often large. In such a situation, use of any common α, e.g. 0·05 or 0·01, will lead often to the choosing of some β at stage one and the assertion that it is significantly different from zero when, in fact, all the β's are zero. How can this be avoided? At one extreme, one can raise the critical level in 4.1. This will lower the error probability we have been discussing but may make the discovery of true non-zero β's very difficult. Is there an optimal way to choose the critical level, as a function of k, to keep all probabilities low, at least for large n?

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A PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATORS FOR INVARIANT STATISTICAL MODELS

BY

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ABSTRACT. This paper generalizes some results on pivotal functions of maximum likelihood estimators of location and scale parameters and the related ancillary statistics obtained by Antle and Bain, and Fisher. It shows that the maximum likelihood estimator of the parameter in an invariant statistical model is an essentially equivariant estimator or a transformation variable in a structural model. In the latter case, ancillary statistics in the sense of Fisher used in conjunction with the maximum likelihood estimators can be easily recognized. It is also remarked that the values of maximum likelihood estimators from samples having the same "complexion" are simply related to those of other, perhaps simpler, transformation variables. In the development it also points out the importance of using the correct definition of the likelihood function originally proposed by Fisher.

Introduction. Recently, Antle and Bain (1969) show that if α and β are the location and scale parameters in the probability density function

$$(1)\quad f(x;\alpha,\beta)=\frac{1}{\beta}g\left(\frac{x-\alpha}{\beta}\right),\qquad -\infty<x<\infty,-\infty<\alpha<\infty,0<\beta<\infty,$$

then the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ have a property described in the following theorem.

THEOREM 1. *$\hat{\beta}_s=\hat{\beta}/\beta$, $\hat{\alpha}_s=(\hat{\alpha}-\alpha)/\beta$ and $\hat{\alpha}_p=(\hat{\alpha}-\alpha)/\hat{\beta}$ are each distributed independent of α and β.*

Applying Theorem 1, Antle, Klimko and Harkness (1970) obtain confidence intervals for the parameters of the logistic distribution.

Similarly, Thoman, Bain and Antle (1969) prove that if α and β are the shape and scale parameters of a two-parameter Weibull distribution whose probability density function is given by

$$(2)\qquad f(x;\alpha,\beta)=(\beta/\alpha)(x/\alpha)^{\beta-1}\exp\{-(x/\alpha)^{\beta}\},\ x>0,\ \alpha>0,\ \beta>0,$$

then the following theorem they give makes it possible to obtain confidence intervals for both α and β.

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THEOREM 2. *If $\hat{\alpha}$ and $\hat{\beta}$ are the maximum likelihood estimators of the Weibull parameters in* (2), *then*

$$\hat{\beta}/\beta \quad \text{and} \quad \beta \ln(\hat{\alpha}/\alpha)$$

are each distributed independent of α and β.

The proofs of both theorems are simple results of a group property of the maximum likelihood estimates inherited from the group structure of the parent models.

The results in Theorem 1 have actually been given earlier by the late Sir R. A. Fisher (1959, p. 163) who observes also that

$$\mu_i = (x_i - \hat{\alpha})/\hat{\beta}, \qquad i = 1, \ldots, N,$$

provide a set of $N-2$ functionally independent ancillary statistics from a sample of size N, whose probability distributions are independent of the parameters and whose values may be said "to specify the complexion of the sample" because the values μ_i will be unchanged if the values x_i we observed had instead the values $X_i = a + cx_i$ for $-\infty < a < \infty$ and $c > 0$. Fisher proposes that in the theory of estimation the precision of the estimates $\hat{\alpha}$ and $\hat{\beta}$ "should be judged solely by reference to the variation of estimates among samples having the same complexion", and that exact probability statements be made about the parameters.

The existence of ancillary statistics which are functions of maximum likelihood estimators for the Weibull distribution (2) can also be easily recognized if we observe that for all constants $a > 0$, and $c > 0$, the maximum likelihood estimates $\hat{\alpha}(X)$ and $\hat{\beta}(X)$ based on a sample $X = (x_1, \ldots, x_N)$ are related to $\hat{\alpha}([a, c]X)$ and $\hat{\beta}([a, c]X)$ based on another sample $[a, c]X$ obtained by a transformation $[a, c]$ of X, where

$$[a, c]X = ([a, c]X_1, \ldots, [a, c]x_N)$$

and

$$[a, c]x_i = a x_i^{1/c}, \qquad i = 1, \ldots, N.$$

The relation is the following:

$$\hat{\alpha}([a, c]X) = a(\hat{\alpha}(X))^{1/c}, \qquad \hat{\beta}([a, c]X) = c\hat{\beta}(X).$$

This is shown by Tan and Sherif (1973).

In this paper we shall give the general result regarding the maximum likelihood estimates for invariant statistical models which contain Theorem 1 and Theorem 2 as special cases or examples.

2. **Main results.**

Let A be a vector random variable defined on an open sample space $S \subset R^N$, θ be a vector parameter with open parameter space Ω which may be taken here as an interval in a Euclidean space R^L, and $f(X; \theta)$ the probability density function of the probability measure P_θ on (S, β) with respect to the Euclidean volume, μ. Let $G = \{g\}$ be a group of one-to-one transformations on S such that $gB \in \beta$ whenever $B \in \beta$, and let $\bar{G} = \{\bar{g}\}$ be the induced group of transformations on Ω defined by the identity

$$P_\theta[B] = P_{\bar{g}\theta}[gB]$$

for every B in β and g in G. We assume that $\bar{g}$ is one-to-one and $\bar{g}\Omega=\Omega$ for all $\bar{g} \in \bar{G}$. The family $\{f(X; \theta):\theta \in \Omega\}$ of densities is then said to specify a statistical model invariant under G.

Given $f(x; \theta)$ we shall define the likelihood function of θ for $X=x$ to be

$$L(\cdot\,; x) = \{kf(x; \cdot):k \in R^{+}\}, \tag{3}$$

which is an equivalence class of similarly shaped functions on Ω involving an arbitrary multiplicative positive constant k. It follows that for each $g \in G$, we have

$$L(\theta; x) = L(\bar{g}\theta; gx), \tag{4}$$

which has been proved by Fraser (1966). We remark that the definition (3) was originally proposed by Fisher in 1922 and is also the current definition of the likelihood function widely used in the literature, but it is not the usual definition found in many introductory texts in mathematical statistics which omits the multiplicative constant k. This incorrect omission of the constant k in the definition of likelihood function would immediately invalidate the result (4) and may have hindered the development of some other results in the theory of statistical inference.

Given the likelihood function $L(\theta; X)$, the maximum likelihood estimator of θ is a (β-measurable) function $\hat{\theta}:S\to\Omega$ satisfying

$$L(\hat{\theta}; X) = \sup_{\theta\in\Omega} L(\theta; X) \quad a.s.\ [\mu].$$

THEOREM 3. *For a statistical model $\{f(x; \theta):\theta \in \Omega\}$ invariant under a group G of transformations, the maximum likelihood estimator $\hat{\theta}$ has the following property:*

$$\hat{\theta}(gX) = \bar{g}\hat{\theta}(X) \tag{5}$$

for all g in G and almost all X in $s[\mu]$.

Zacks (1971, p. 319) calls such an estimator satisfying condition (5) an essentially equivariant estimator.

Proof.

$$\begin{aligned}\sup_{\theta\in\Omega} L(\theta; gX) &= L(\hat{\theta}(gX); gX)\\ &= L(g^{-1}\hat{\theta}(gX); X) \le \sup_{\theta\in\Omega} L(\theta; X)\\ &= L(\hat{\theta}(X); X) = L(\bar{g}\hat{\theta}(X); gX)\\ &\le \sup_{\theta\in\Omega} L(\theta; gX).\end{aligned}$$

Hence (5) follows.

If the invariant statistical model $\{f(x; \theta):\theta \in \Omega\}$ is also a structural model in the sense of Fraser (1968, p. 49) so that the random variable X and the parameter θ

are related to an error variable E with a known distribution the sample space S by the structural equation

$$X = \theta E,$$

where θ takes values in a unitary group G of transformations of S onto S, then the maximum likelihood estimator $\hat{\theta}(X)$, denoted by $[X]$ for convenience, is a transformation variable [Fraser, 1968, p. 51] taking values in G and

$$[gX] = g[X] \quad \text{for all } g \text{ in } G. \tag{6}$$

COROLLARY. *If $f(X;\theta)$ is a structural model, then the function $\theta^{-1}\hat{\theta}(X)$ is a pivotal quantity and $D(X)=[x]^{-1}X$ is a set of ancillary statistics whose probability distributions are independent of the parameter θ.*

Proof.

(*i*) $\theta^{-1}\hat{\theta}(X) = [\theta^{-1}X] = [E] = \hat{\theta}(E)$;

(*ii*) $D(X) = [X]^{-1}X = [E]^{-1}E.$

Since $D(gX)=[X]^{-1}X$ for all g in G, D is the "complexion" of the sample conditional on which, according to Fisher, the precision of $\hat{\theta}(X)$ should be judged.

REMARKS. While the above results show that for a structural model the maximum likelihood estimate $\hat{\theta}(X)$, if it exists, is a transformation variable $[X]$, satisfying the relation (6), often there are other transformation variables satisfying the same relation. Suppose $[X]_1=T(X)$ is a statistic taking values in G and satisfying (6). Then $D_1(X)=[X]_1^{-1}X$ is a set of ancillary statistics. We see then there exists a unique g in G such that

$$D(X) = gD_1(X), \tag{7a}$$

$$[X] = [X]_1 g^{-1} \quad \text{or} \quad \hat{\theta}(X) = T(X)g^{-1}, \tag{7b}$$

where $g=T(D(X))$. Thus in cases where it is difficult to obtain numerical values of maximum likelihood estimates $\hat{\theta}(X)$ but relatively easy to calculate $T(X)$, for repeated samples having the same "complexion" $D(X)$, we may obtain $\hat{\theta}(X)$ from $T(X)$ by (7b). Simple examples of this are given by Tan and Drossos (1973).

In structural inference, the structural distribution of θ obtained from the conditional distribution of $\hat{\theta}(E)$ given $D(X)$ and that obtained from the conditional distribution of $T(E)$ given $D_1(X)$ are equivalent [Fraser, 1968, p. 32].

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PARTIAL ORDERS ON THE 2-CELL

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1. **Introduction.** A *partially ordered space* is an ordered pair $(X, \leq)$ where X is a compact metric space and $\leq$ is a partial ordering on X such that $\leq$ is a closed subset of the Cartesian product $X \times X$. $\leq$ is said to be a *closed partial order* on X.

If $(X, \leq)$ is a partially ordered space let $\text{Min}(X)$ (resp. $\text{Max}(X)$) denote the set of minimal (resp. maximal) elements of X. For $x \in X$ let

$$L(x) = \{y \in X \mid y \leq x\} \quad \text{and} \quad M(x) = \{y \in X \mid x \leq y\}.$$

Ward used partial orders to characterize dendrites in [7]. In [1], [2] and [3] Tymchatyn used partial orders to obtain characterizations of the two and three dimensional cells.

In this paper we use the methods developed in [1] to study a wide class of partial orders on the 2-cell. We let $(X, \leq)$ be a partially ordered space where X is a 2-cell, $\text{Min}(X)$ and $\text{Max}(X)$ are closed arcs on the boundary of X and for each $x \in X$ $L(x) \cup M(x)$ is a connected set. We let $\leq'$ be the vertical partial order on the unit square $[0, 1] \times [0, 1]$ in the euclidean plane, i.e. we set $(a, b) \leq' (c, d)$ in $[0, 1] \times [0, 1]$ if and only if $a = c$ and $d - b$ is non-negative. We show that there is a continuous order preserving function of the partially ordered space $([0, 1] \times [0, 1], \leq')$ onto $(X, \leq)$ such that the inverse image of a point of X is either a point or a horizontal line segment. It follows that $\leq$ contains a partial order that has the same properties as $\leq$ and that is obtainable in a natural way from a very simple decomposition of $([0, 1] \times [0, 1], \leq')$.

2. **Preliminaries.** We shall gather here some necessary definitions and theorems from [1] and [4].

A *chain* is a totally ordered set. An *order arc* is a compact connected chain. It is known [6] that a separable order arc is homeomorphic under an order preserving function to the closed unit interval [0, 1] with its usual order (which we also denote by $\leq$) and with its usual topology. The reader should have no difficulty in determining in a particular instance whether $\leq$ represents the partial order on X or on [0, 1].

If $(X, \leq)$ is a partially ordered space we let 2^X denote the space of closed subsets of X with the Hausdorff metric topology. We let $\mathscr{M}(X)$ denote the family of order arcs in X which meet both $\text{Min}(X)$ and $\text{Max}(X)$.

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THEOREM A (TYMCHATYN and WARD [4]). *Let* $(X, \leq)$ *be a partially ordered space such that* $\mathrm{Min}(X)$ *and* $\mathrm{Max}(X)$ *are closed sets and for each* $x \in X$, $L(x) \cup M(x)$ *is connected. Then* $\mathscr{M}(X)$ *is a compact subset of* 2^X *and* $\mathscr{M}(X)$ *covers* X.

An *antichain* is a set which contains no non-degenerate chain. We let $\mathscr{A}(X)$ denote the family of compact maximal antichains of X. We let $\mathscr{A}(X)$ have its relative topology as a subset of 2^X. It is known [6] that $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ are in $\mathscr{A}(X)$ if and only if they are closed subsets of X.

The following two results appear in [1]:

THEOREM B. *Let* $(X, \leq)$ *be a partially ordered space such that* $Min(X)$ *and* $Max(X)$ *are closed and* $\mathscr{M}(X)$ *covers* X. *For* $A, B \in \mathscr{A}(X)$ *define*

$$A \wedge B = \{x \in A \cup B \mid L(x) \cap (A \cup B) = \{x\}\}$$

and

$$A \vee B = \{x \in A \cup B \mid M(x) \cap (A \cup B) = \{x\}\}.$$

Then $\mathscr{A}(X)$ *with operations* $\wedge$ *and* $\vee$ *is an arcwise connected topological lattice. Furthermore,* $\mathscr{A}(X)$ *covers* X.

THEOREM C. *Let* $(X, \leq)$ *be a partially ordered space such that* $\mathrm{Max}(X)$ *and* $\mathrm{Min}(X)$ *are closed, disjoint sets and* $\mathscr{M}(X)$ *covers* X. *Then there exists a continuous order preserving function* f *of* $(X, \leq)$ *onto* $[0, 1]$ *with its usual order such that*

(i) $f^{-1}(0) = \mathrm{Min}(X)$ *and* $f^{-1}(1) = \mathrm{Max}(X)$ *and*
(ii) *for each* $a \in [0, 1]$ $f^{-1}(a) \in \mathscr{A}(X)$.

The partial order $\leq$ on a space X is said to be *order dense* if for each $x<y$ there exists z such that $x<z<y$. It is known [6] that if $(X, \leq)$ is a compact, order dense, partially ordered space then every chain in X is contained in a member of $\mathscr{M}(X)$.

In case there is more than one partial order on a space X we shall write $\mathrm{Min}(X, \leq)$, $\mathrm{Max}(X, \leq)$, $L(x, \leq)$, $M(x, \leq)$, $\mathscr{M}(X, \leq)$ and $\mathscr{A}(X, \leq)$ for $\mathrm{Min}(X)$, $\mathrm{Max}(X)$, $L(x)$, $M(x)$, $\mathscr{M}(X)$ and $\mathscr{A}(X)$ respectively.

3. **Orders on the 2-cell.**

THEOREM 1. *Let* $(X, \leq)$ *be a partially ordered space where* X *is a closed* 2-*cell. Suppose* $\mathrm{Min}(X)$ *and* $\mathrm{Max}(X)$ *are arcs in the boundary* S^1 *of* X. *If* $\mathscr{M}(X)$ *covers* X *then* $\mathscr{M}(X)$ *admits the structure of a compact, connected, topological lattice.*

Proof. Case 1. Suppose $\mathrm{Min}(X)$ is disjoint from $\mathrm{Max}(X)$.

Let F and E be the closures of the two components of $S^1 - (\mathrm{Min}(X) \cup \mathrm{Max}(X))$. We wish to prove that F and E are in $\mathscr{M}(X)$. Notice that F and E are arcs which meet both $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$. We need only prove that F and E are chains. Let $x, y \in F$. We may suppose that x separates y from $\mathrm{Min}(X) \cap F$ in F. By hypothesis there exists $T \in \mathscr{M}(X)$ such that $x \in T$. If $y \in T$ we are done. If $y \notin T$ then

$T \cap M(x)$ is an arc in the 2-cell X which separates y from $\mathrm{Min}(X) \cap L(y)$. By hypothesis there exists $S \in \mathscr{M}(X)$ such that $y \in S$. Now $S \cap L(y)$ is an order arc which meets both y and $\mathrm{Min}(X) \cap L(y)$. Hence, there exists

$$z \in S \cap L(y) \cap T \cap M(x).$$

Since $x \leq z \leq y$ it follows that $x \leq y$ and F is a chain. Similarly, E is a chain.

By Theorem C there is a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) = \mathrm{Min}(X)$, $f^{-1}(1) = \mathrm{Max}(X)$ and for each $a \in [0, 1]$, $f^{-1}(a) \in \mathscr{A}(X)$. It follows that for each $a \in [0, 1]$, $f^{-1}(a)$ meets each member of $\mathscr{M}(X)$ in precisely one point.

If $a \in]0, 1[$, it is clear that $f^{-1}(a)$ separates X into precisely two components $f^{-1}([0, a[)$ and $f^{-1}(]a, 1])$. Furthermore, $f^{-1}(a)$ is arcwise accessible from each of these two components. By Theorem II.5.38 in Wilder [8], $f^{-1}(a)$ is an arc. Since $f^{-1}(a)$ is irreducible with respect to separating X it follows that one endpoint of $f^{-1}(a)$ is in F and the other is in E.

For each $a \in [0, 1]$ give the arc $f^{-1}(a)$ its natural total order with minimal point in $F \cap f^{-1}(a)$. Denote the resulting partial order on X by $\leq''$. Then $x \leq'' y$ if and only if $f(x) = f(y)$ and x separates y from $f^{-1}(f(x)) \cap F$ in the arc $f^{-1}(f(x))$.

CLAIM. *$(X, \leq'')$ is a partially ordered space.*

Proof of claim. Let $x_i)$ and $y_i)$ be sequences in X which converge to x and y respectively such that for each i, $x_i \leq'' y_i$. We must prove that $x \leq'' y$. For each i, $f(x_i) = f(y_i)$ and f is continuous so $f(x) = f(y)$. Either $x \leq'' y$ or $y \leq'' x$ since $f^{-1}(f(x))$ is linearly ordered with respect to the partial order $\leq''$. Just suppose $y <'' x$. Let $t \in X$ such that $y <'' t <'' x$. By hypothesis there exists $T \in \mathscr{M}(X, \leq)$ such that $t \in T$. The arc T separates x and y in X. The sequence $x_i)$ (resp. $y_i)$) is eventually in the same component of $X - T$ as is x (resp. y) since X is locally connected. Also, $F \cap f^{-1}(f(x_i))$ is eventually in the same component of $X - T$ as $F \cap f^{-1}(f(x))$ and y. Since for each i, $f^{-1}(f(x_i))$ meets T in precisely one point, it follows that eventually $y_i <'' x_i$. This is a contradiction. Thus $x \leq'' y$ and the claim is proved.

Clearly, $\mathscr{M}(X, \leq) \subset \mathscr{A}(X, \leq'')$. By Theorem A $\mathscr{M}(X, \leq)$ is compact. By Theorem B $\mathscr{A}(X, \leq'')$ is a topological lattice. We must prove that $\mathscr{M}(X, \leq)$ is actually a sublattice of $\mathscr{A}(X, \leq'')$. If $S, T \in \mathscr{M}(X, \leq)$ then $T \wedge S$ and $T \vee S$ are in $\mathscr{A}(X, \leq'')$ since $\mathscr{A}(X, \leq'')$ is a lattice. In particular $T \vee S$ and $T \wedge S$ are compact subsets of X. The continuous function f takes each of the compact sets $T \wedge S$ and $T \vee S$ by a one-to-one correspondence onto $[0, 1]$. Hence, $T \wedge S$ and $T \vee S$ are arcs. We check that $T \wedge S$ and $T \vee S$ are chains with respect to $\leq$. Let $x, y \in T \wedge S$. We may suppose that $f(x) < f(y)$. If $x, y \in T$ then $x \leq y$ since T is a chain. Suppose, therefore, that $x \in S - T$ and $y \in T - S$. Let z be the maximal element of the compact chain

$$(T \wedge S) \cap S \cap f^{-1}([0, f(y)]).$$

Then $z \in T \cap S$ since $f^{-1}(f(z)) \cap (T \wedge S)=\{z\}$, $T \wedge S$ is compact and

$$(T \wedge S) \cap f^{-1}([f(z), f(y)]) \subset T.$$

Hence, $x \leq z \leq y$ and $T \wedge S$ is a chain with respect to $\leq$. Since $T \wedge S$ is a compact, connected chain which meets both $\mathrm{Min}(X, \leq)$ and $\mathrm{Max}(X, \leq)$ it follows that $T \wedge S \in \mathscr{M}(X, \leq)$. Similarly, $T \vee S \in \mathscr{M}(X, \leq)$.

For $T, S \in \mathscr{M}(X, \leq)$ define $T \leq^* S$ if and only if $T \wedge S=T$. Then $(\mathscr{M}(X, \leq), \leq^*)$ is a partially ordered space. We shall prove that $\leq^*$ is order dense. Let $S, T \in \mathscr{M}(X, \leq)$ such that $T \wedge S=S$ and $S \neq T$. Let $a \in [0, 1]$ such that $f^{-1}(a) \cap S \neq f^{-1}(a) \cap T$ and let $b \in f^{-1}(a)$ such that

$$f^{-1}(a) \cap S <'' b <'' f^{-1}(a) \cap T.$$

By hypothesis there exists $P \in \mathscr{M}(X, \leq)$ such that $b \in P$. Let $R=(S \vee P) \wedge T$. Then $b \in R \in \mathscr{M}(X, \leq)$. Notice that $T \wedge R=R$ and $R \wedge S=S$. Hence, $S <^* P <^* T$. Thus, $\leq^*$ is order dense. Since F is the unique minimal element in the partially ordered space $(\mathscr{M}(X, \leq), \leq^*)$, $\mathscr{M}(X, \leq^*)$ is connected by the remarks following Theorem C.

Case 2. Suppose $\mathrm{Min}(X, \leq) \cap \mathrm{Max}(X, \leq)$ is non-void. Make the disjoint union of X and $\mathrm{Max}(X, \leq) \times [0, 1]$ into a partially ordered space by setting $x \leq' y$ in $X \cup (\mathrm{Max}(X, \leq) \times [0, 1])$, if:

(a) $x, y \in X$ and $x \leq y$
(b) $x \in X$ and $y=(a, b) \in \mathrm{Max}(X, \leq) \times [0, 1]$ where $x \leq a$ in X or
(c) $x=(a, b)$ and $y=(c, d)$ are in $\mathrm{Max}(X, \leq) \times [0, 1]$, $a=c$ and $b \leq d$ in $[0, 1]$.

Form the adjunction space X' of X and $\mathrm{Max}(X, \leq) \times [0,1]$ by identifying (m, o) and m for each $m \in \mathrm{Max}(X, \leq)$. The partial order $\leq'$ on $X \cup (\mathrm{Max}(X, \leq) \times [0, 1]$ induces a partial order $\leq^\circ$ on X' such that $(X', \leq^\circ)$ satisfies the hypotheses of the theorem. Notice that $\mathrm{Min}(X', \leq^\circ)=\mathrm{Min}(X, \leq)$ and $\mathrm{Max}(X', \leq^\circ)=\mathrm{Max}(X, \leq) \times \{1\}$. By Case 1 $\mathscr{M}(X', \leq^\circ)$ is a compact, connected topological lattice. It is easy to see that $\mathscr{M}(X, \leq)$ is homeomorphic and isomorphic to $\mathscr{M}(X', \leq^\circ)$ under the correspondence that takes a member A of $\mathscr{M}(X', \leq^\circ)$ to the unique member B of $\mathscr{M}(X, \leq)$ such that $B \subset A$.

The following theorem was proved in [1].

THEOREM D. *Let* $(X, \leq)$ *be a partially ordered space. Suppose there exists a function* $h: [0, 1] \rightarrow \mathscr{M}(X)$ *such that* $X=\cup\{h(a) \mid a \in [0, 1]\}$ *and if* $a<b<c$ *in* $[0, 1]$ *then* $h(a) \cap h(c) \subset h(b)$. *If* X *is non-degenerate and has no cutpoints then* X *is a 2-cell.*

It is shown in the proof of Theorem D that if $\leq$ is a closed partial order on the 2-cell D then in order that there exist a function $h: [0, 1] \rightarrow \mathscr{M}(D)$ as in the above theorem it is necessary that $\mathrm{Min}(D)$ and $\mathrm{Max}(D)$ be closed, connected sets in the boundary of D and that $\mathscr{M}(D)$ cover D. We shall show that these conditions are also sufficient.

Corollary 2. *Let $\leq$ be a closed partial order on a 2-cell D such that $\mathscr{M}(D)$ covers D and* Min(D) *and* Max(D) *are closed and connected sets in the boundary of D. Then there exists a continuous function $h:[0, 1]\to\mathscr{M}(D)$ such that $D=\cup\{h(\mathrm{a}) \mid a\in[0, 1]\}$ and if $a<b<c$ in* $[0, 1]$ *then $h(a)\cap h(c)\subset h(b)$.*

Proof. $\mathscr{M}(D)$ is a compact topological lattice and thus $\mathscr{M}(D)$ has a zero F and a unit E. By Theorem 1 $\mathscr{M}(D)$ is connected. By Koch's Theorem (see [5]) there is an order arc $\mathbb{C}$ in $\mathscr{M}(D)$ such that $F, E\in\mathbb{C}$. Let $h:[0, 1]\to\mathbb{C}$ be a one to one continuous function such that $h(0)=F$ and $h(1)=E$. From the definition of order in $\mathscr{M}(D)$ it is clear that if $a<b<c$ in $[0, 1]$ then $h(a)\cap h(c)\subset h(b)$. It remains to show only that $D=\cup\{h(a)\mid a\in[0, 1]\}$.

Let $x\in D$ and let $R\in\mathscr{A}(D)$ such that $x\in R$ by Theorem B. It is easy to see that for each $a\in[0, 1]$, $h(a)\cap R$ consists of exactly one point. Define $g:[0, 1]\to R$ by letting $g(a)\in h(a)\cap R$ for each $a\in[0, 1]$. Then g is easily seen to be a continuous function. By the proof of Theorem 1 R is an arc with endpoints in F and E. Hence $g(0)\in F$ and $g(1)\in E$. Thus, g maps $[0, 1]$ onto R and $x\in\cup\{h(a) \mid a\in [0, 1]\}$.

Let $(D', \leq')$ be the unit square $[0, 1]\times[0, 1]$ in the plane with the partial order $(a, b)\leq'(c, d)$ if and only if $a=c$ and $b\leq d$ in $[1, 1]$.

Corollary 3. *Let $(D, \leq)$ be a partially ordered space such that D is a 2-cell, $\mathscr{M}(D, \leq)$ covers D and* Min$(D, \leq)$ *and* Max$(D, \leq)$ *are closed disjoint arcs in the boundary of D. There is an order preserving continuous function g of $(D', \leq')$ onto $(D, \leq)$ such that*

(i) $g^{-1}(\mathrm{Max}(D, \leq))=\mathrm{Max}(D', \leq')$

(ii) $g^{-1}(\mathrm{Min}(D, \leq))=\mathrm{Min}(D', \leq')$

(iii) *g takes every member of $\mathscr{M}(D', \leq')$ homeomorphically onto a member of $\mathscr{M}(D, \leq)$.*

(iv) *for $x\in D$ $g^{-1}(x)$ is either a point or a horizontal line segment in D'.*

Proof. Let $f:D\to[0, 1]$ be a function satisfying the conditions of Theorem C. Let $h:[0, 1]\to\mathscr{M}(D, \leq)$ be a function satisfying the conditions of Corollary 2. Define $g:(D', \leq')\to(D, \leq)$ by letting $g(a, b)$ be the unique point in $h(a)\cap f^{-1}(b)$ for each $(a, b)\in D'=[0, 1]\times[0, 1]$.

If $(D, \leq)$ and g are as in Corollary 3, there is a smallest partial order $\leq^*$ on D such that g is order preserving with respect to $\leq^*$ and $\leq^*$ has a closed graph. Clearly $\leq^*\subset\leq$, $\mathscr{M}(D, \leq^*)$ covers D, $\mathrm{Min}(D, \leq^*)=\mathrm{Min}(D, \leq)$ and $\mathrm{Max}(D, \leq^*)=\mathrm{Max}(D, \leq)$. Thus we have extracted from $\leq$ a partial order $\leq^*$ which is moderately large and which is well understood since it is completely determined by the function g.

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A UNIQUENESS THEOREM ON THE DEGENERATE CAUCHY PROBLEM(1)

BY

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In [4] Carroll and the author have treated the following problem

$$u_{tt}+\Lambda^{\alpha}S(t)u_t+\Lambda^{\beta}R(t)u+\Lambda a(t)u+B(t)u=f(t) \tag{1}$$

where Λ is a closed densely defined self-adjoint operator in a separable Hilbert space H with $(\Lambda u, u)\geq c\,\|u\|^2$, $c>0$, $\Lambda^{-1}\in L(H)$ ($L(E, F)$ is the space of continuous linear maps from E to F; in particular, $L(H)=L(H, H)$), $a(t)>0$ for $t>0$, $a(0)=0$ and $S(t)$, $R(t)$, $B(t)\in L(H)$. Various other hypotheses will be given later. It is assumed that all operators commute, and we seek $u\in\mathscr{E}^2(H)$ ($\mathscr{E}^m(H)$ is the space of m-times continuously differentiable functions of t with values in H) satisfying (1) with

$$u(0)=u_t(0)=0. \tag{2}$$

Existence and uniqueness theorems have been obtained for the problem (1)–(2) in [4]. The purpose of this paper is to prove a somewhat stronger uniqueness theorem for the problem. We also give examples that will be provided for differential equations to which the uniqueness theorem applies but which are not included in any previously known results (e.g. see [1, 4, 6, 8, 10]). A counter-example is also given in order to indicate that the function of degeneracy $a(t)$ can only admit oscillations under a suitable condition.

Notations and symbols in [4] are adopted here except for arbitrary constants. Indeed equations taken from [4] will be renumbered in order to make our paper as self-contained as possible.

In order to apply spectral methods we set that $S(t)=Ss(t)$, $R(t)=Rr(t)$, $B(t)=Bb(t)$, where B, R, S, Λ^{-1} commute and are bounded normal with $b, r, s\in c^0[0, l]$, $a\in c^1[0, l]$. Let $\mathscr{A}$ be the uniformly closed $*$ algebra generated by Λ^{-1}, B, R, S, B^*, R^*, S^* and I. Then we associate with these operators (other than I) the complex spectral variables $z_0, z_1, \dots, z_6$ (see [3, 4]), $z_0=1/\lambda$ is real. Now the map $m:\Phi_{\mathscr{A}}\to\mathbb{C}^7$ given by $m(\phi)=(\hat{\Lambda}^{-1}(\phi), \hat{B}(\phi), \dots, \hat{S}^*(\phi))$ is a homeomorphism of the carrier space $\Phi_{\mathscr{A}}$ with the joint spectrum σ of the elements $\Lambda^{-1}, B, \dots, S^*$ (see [4, 9]). We consider now in connection with (1) the homogeneous equation

$$u_{tt}+\lambda^{\alpha}z_3s(t)u_t+\lambda^{\beta}z_2r(t)u+\lambda a(t)u+z_1b(t)u=0. \tag{3}$$

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Solutions $Z(t, \tau, z_i, \lambda)$ and $Y(t, \tau, z_i, \lambda)$ of (3) with $Z(\tau, \tau)=1$, $Z_t(\tau, \tau)=0$, $Y(\tau, \tau)=0$, $Y_t(\tau, \tau)=1$ (cf. [2]) will give rise to operators in the von Neumann algebra $\mathscr{A}''$ if Y and Z are continuous in (z_i, λ) for $|z_i|\leq c_1$ $(i=1, \ldots, 6)$, $|\lambda|\leq R_0$ (R_0 arbitrary), and bounded for $|z_i|\leq c_1$, $|z_0|\leq 1/c$ (this has been proved in [3]). The constant c_1 is chosen so that $c_1\geq\max\{\|B\|, \|R\|, \|S\|\}$ and then the joint spectrum σ lies within the region $|z_i|\leq c_1$ $(i=1, \ldots, 6)$, $|z_0|\leq 1/c$ (note that $\lambda\to\infty$ corresponds to $z_0\to 0$).

It is known from classical results (see [5, 6]) that for $0\leq\tau\leq t\leq l<\infty$ there exist unique Z and Y as required, continuous in (t, τ, z_i, λ) in the region $0\leq\tau\leq t\leq l<\infty$, $|z_i|\leq c_1$ $(i=1, \ldots, 6)$, $0<z_0\leq 1/c$. However, Y and Z may not be analytic single-valued in z_i, λ, because α, β may be fractional. Thus by introducing the Green's operator associated with (3) (see [4, 10] for details) we obtain formally for the solution of (1)–(2):

$$u(t) = \int_\tau^t \mathscr{Y}(t, \xi) f(\xi)\, d\xi \tag{4}$$

where $\mathscr{Y}$ is the operator associating with Y.

For convenience, we now summarize what have been assumed (in [4]): (i) $a\in c^1[0, l]$, (ii) $s, r, b\in c^0[0, l]$, (iii) $P=a'+c_1(r^2+b^2/\lambda)\geq 0$, (iv) $\gamma=\max\{\alpha, \frac{1}{2}\}$, (v) $2\beta\leq 1$, (vi) $\operatorname{Re}(z_3 s(t))\geq 0$, (vii) $f/Q=h\in\mathscr{E}^0(D(\Lambda^\gamma))$ $(Q=a^{1/2}q=(a\phi\psi)^{1/2}$, with $\phi(\tau)=\exp(-c_1\int_\tau^l r^2/a d\xi)$, $\psi(\tau)=\exp(-(c_1/\lambda)\int_\tau^l b^2/a d\xi)$.

According to [4], the solution of (1) with values $u(\tau)$ and $u_t(\tau)$ prescribed, $0<\tau\leq t$, can be written as follows:

$$u(t) = \mathscr{Z}(t, \tau)u(\tau)+\mathscr{Y}(t, \tau)u_t(\tau)+\int_\tau^t \mathscr{Z}(t, \xi)f(\xi)\, d\xi, \tag{5}$$

where $\mathscr{Z}$, $\mathscr{Y}$ are operators associating with Z, Y which are solutions of (3).

We recall that two uniqueness theorems were proved in [4]. One of them required strongly $u\in\mathscr{E}^2(H)$, $u/q\in\mathscr{E}^0(D(\Lambda^{\gamma+1/2}))$ and $u_t/Q\in\mathscr{E}^0(D(\Lambda^\gamma))$. On the other hand if $q>0$ a second uniqueness theorem had been given as Theorem 3 in page 254 of [4]. We now state it without proof as follows:

Theorem 3 of [4]. Assume u is a solution of (1)–(2) with $u\in\mathscr{E}^0(D(\Lambda^{\gamma+1/2}))\cap\mathscr{E}^1(D(\Lambda^\gamma))\cap\mathscr{E}^2(H)$ and let $\int r^2/a d\xi<\infty$, $\int b^2/a d\xi<\infty$ with $(\int_0^t a(\xi)\, d\xi)^\delta/a^{1/2}(t)$ continuous for some $\delta<\frac{1}{2}$. Then u is unique.

It is noted that the second uniqueness theorem of [4] stated above was based upon the following inequality:

$$\|u_t\| \leq c_2\left(\int_0^t a(\xi)\, d\xi\right)^{1/2}. \tag{6}$$

On recalls that the degenerate Cauchy Problems of a similar form (e.g. see [7, 8]) were solved with $a(t)$ monotone (monotonicity of $a(t)$ is not necessarily required here or in [1, 4, 10]). However, from (6) with a monotone $a(t)$, it follows immediately that ($t\leq l<\infty$)

$$\|u_t\| \leq c_3 a^{1/2}(t). \tag{7}$$

This means that $\|u_t/a^{1/2}\|$ is bounded for every $t\in[0,l]$. Now $(t,\tau)\to a^{1/2}(\tau)\mathscr{Y}(t,\tau)$ is continuous with values in $L_S(H, D(\Lambda^{1/2}))$ (where the subscript S denotes the strong operator topology), and the term $\mathscr{Y}(t,\tau)u_t(\tau)$ in (5) may be written as

$$\mathscr{Y}(t,\tau)u_t(\tau) = a^{1/2}(\tau)\mathscr{Y}(t,\tau)\,\frac{u_t(\tau)}{a^{1/2}(\tau)}\,. \tag{8}$$

But $u_t(\tau)/a^{1/2}(\tau)$ is also continuous. Moreover, since $q>0$ $\mathscr{X}(t,\tau)u(\tau)\to 0$ as $\tau\to 0$. Actually, the above arguments would remain true if $a(t)$ were required to be locally monotone (i.e., $a(t)$ is monotone in a neighborhood of 0). Oscillations in $a(t)$ are then allowed.

In order to facilitate the subsequent arguments, we rewrite (6) as

$$\|u_t/a^{1/2}\| \leq c_2 A^{1/2}(t) \tag{9}$$

where

$$A(t) = \int_0^t a(\xi)\,d\xi/a(t). \tag{10}$$

It is easily seen that the boundedness of $A(t)$ on $[0,l]$ would assure uniqueness for the problem (1)–(2) in view of (8). In fact, if $a(t)$ is assumed to be locally monotone, $A(t)$ is obviously bounded on $[0,l]$. Consequently, we simply replace the condition $(\int_0^t a(\xi)\,d\xi)^\delta/a^{1/2}(t)$ being continuous for some $\delta<\frac{1}{2}$ by $a(t)$ being locally monotone in Theorem 3 of [4]. Thus we have

THEOREM. *Assume* (i), (ii), (iii), (iv), (v), (vi), (vii). *Let u be a solution of* (1)–(2) *with* $u\in\mathscr{E}^0(D(\Lambda^{\gamma+1/2}))\cap\mathscr{E}^1(D(\Lambda^{\gamma}))\cap\mathscr{E}^2(H)$, $\int r^2/a\,d\xi<\infty$, $\int b^2/a\,d\xi<\infty$. *If $a(t)$ is locally monotone, then u is unique.*

It may be worthwhile to point out that in case $a'(0)\neq 0$ or $a(t)\sim kt^n$ (i.e., $\lim_{t\to 0} a(t)/t^n=k>0$ for $n>0$), $A(t)$ is evidently bounded. Indeed both cases are special cases of the theorem. For $a(t)$ being locally monotone, some class of nonmonotone functions $a(t)$ (such as $a(t)=\exp(-1/t)\;[a_1+a_2t^3\sin^2(1/t)]$, $a(t)=a_1t^n+a_2t^p\sin(1/t)$ with $a_1\geq|a_2|, p>n>0, p>2$) would be included.

Finally, it is not hard to see that for a nonmonotone $a(t)=O(t^n)$ with $n>0$, $A(t)$ may not be bounded; in particular, $\limsup A(t)=\infty$. To this end, consider* the function $a_1(t)$ and $a_2(t)$ defined on $[0, 1]$ as follows:

$$a_1(t) = t^3\sin^2(1/t)+t^5\cos^2(1/t),$$
$$a_2(t) = t^3\cos^2(1/t)+t^5\sin^2(1/t).$$

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* Professor A. Meir constructed these functions for the author when he attended the 1967 Summer Research Institute in Edmonton.

Clearly,

$$a_i(t) \leq t^3, \qquad i = 1, 2 \tag{12}$$

and

$$a_1(t)+a_2(t) = t^3+t^5, \tag{13}$$

for all $t \in [0, 1]$. (13) implies that

$$\int_0^t \{a_1(\xi)+a_2(\xi)\}\, d\xi > \int_0^t \xi^3\, d\xi = t^4/4$$

for all $t \in [0, 1]$ and consequently

$$\int_0^t a_1(\xi)\, d\xi > t^4/8 \tag{14}$$

or

$$\int_0^t a_2(\xi)\, d\xi > t^4/8 \tag{15}$$

or both hold for infinitely many $t \in (0, 1]$. On the other hand it follows from (12) that

$$\int_0^t a_i(\xi)\, d\xi \leq t^4/4, \qquad i = 1, 2$$

for all $t \in [0, 1]$. Now consider two particular sequences $\{t_k\}$, $\{t_k'\}$ of $(0, 1)$ where

$$t_k = 1/2k\pi, \qquad t_k' = 1/(2k-\tfrac{1}{2})\pi, \qquad k = 1, 2, \ldots .$$

(For simplicity, we use the same notations for their subsequences if applicable.)

Evidently,

$$a_1(t_k) = t_k^5, \qquad a_1(t_k') = t_k'^3;$$
$$a_2(t_k) = t_k^3, \qquad a_2(t_k') = t_k'^5.$$

Thus we obtain

$$\frac{\int_0^{t_k'} a_1(\xi)\, d\xi}{a_1(t_k')} \leq \frac{t_k'^4/4}{t_k'^3} = t_k'/4 \tag{16}$$

and

$$\frac{\int_0^{t_k} a_2(\xi)\, d\xi}{a_2(t_k)} \leq \frac{t_k^4/4}{t_k^3} = t_k/4. \tag{17}$$

Now if (14) holds for infinitely many $t \in (0, 1]$ and in particular $\{t_k\}$ is considered, then

$$\frac{\int_0^{t_k} a_1(\xi)\, d\xi}{a_1(t_k)} > \frac{t_k^4/8}{t_k^5} = 1/8t_k. \tag{18}$$

Otherwise, (15) must hold for $\{t'_k\}$ so that we have

$$(19)\quad \frac{\int_0^{t_k} a_2(\xi)\,d\xi}{a_2(t'_k)} > \frac{\int_0^{t_k} a_2(\xi)\,d\xi}{t'^5_k} > \frac{t_k^4/8}{t'^5_k} = (t_k/t'_k)^4/8t'_k = [(2k-\tfrac{1}{2})/2k]^4/8t'_k \geq (3/4)^4/8t'_k$$

Noting (16) (or (17)) and (18) (or (19)), we conclude that $A(t)$ may be unbounded even if $a(t)=0(t^n)$ for some $n>0$.

The above counter-example shows that $a(t)$ can only admit oscillations conditionally. In the present case $A(t)$ must be bounded on $[0, l]$.

REMARK. The theorem may be regarded as somewhat "best possible" in the sense that the case $a(t)\sim t^n$(for $n>0$) is a special case of the theorem while the case $a(t)=0(t^n)$ (for $n>0$) cannot be generally included in the theorem (see counter-example above).

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A NOTE ON RADICAL EXTENSIONS OF RINGS

BY

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All rings are associative. A ring T is said to be radical over a subring R if for every $t \in T$, there exists a natural number $n(t)$ such that $t^{n(t)} \in R$.

In [1] Faith showed that if T is radical over R and T is primitive, then R is primitive. We might then ask if the same is true if prime is substituted for primitive. This is not in general true if T does not have a unity element or if char $T \neq 0$. However, we do have

THEOREM 1. *Suppose T is radical over R, T and R have a unity element*, char $T=0$, *and T is prime. Then R is prime.*

The above theorem follows easily from the following

THEOREM 2. *Suppose that the ring T is radical over a subring R, R and T have a common unity element, and T is torsion-free as a Z-module. Then $T_{Z^*}=R_{Z^*}$, where R_{Z^*} is the localization of R at the nonzero integers.*

In proving theorem 2, we use the following

THEOREM 3. (*Kaplansky* [2]) *Suppose that a field K is radical over a proper subfield F. Then K has prime characteristic, and is either purely inseparable over F, or algebraic over its prime subfield.*

Proof of theorem 2. We prove the theorem by assuming that T and R are Q-algebras, and showing that $R=T$.

We first show that every nilpotent element of T lies in R. Suppose $x \in T\backslash R$ is nilpotent. From the sequence $x, x^2, x^3, \ldots$, choose k maximal such that $x^k \notin R$. Since T is radical over R, $\exists\, n$ such that $(1+x^k)^n \in R$. Then $1+nx^k+\cdots+x^{kn} \in R$, from which we deduce that $x^k \in R$, a contradiction.

Now suppose that T is a commutative Artinian Q-algebra and that R is also Artinian. Since the Jacobson radical of an Artinian ring is nilpotent, we have $J(T)=J(R)=J$. T/J is a finite direct product of fields, and is radical over R/J. By Kaplansky's theorem $T/J=R/J$, hence $T=R$.

Now let T be arbitrary. Suppose $x \in T$. Then $Q[x]=A$ is radical over $Q[x] \cap R=B$. If A is finite dimensional, then $A=B$, by the above result. If x is transcendental over Q, localize A at B^*, the nonzero elements of B. Since A is radical over B, A_{B^*} is a field, radical over the field B_{B^*}. Hence, once again, $A_{B^*}=B_{B^*}$. Take $r \in B$, $r\neq 0$, such that $rx \in B$, and let $s=r^n$, where $x^n \in B$. We now have $sA \subset B$. Let $I=(s)$, and note A/I is radical over B/I. However, A/I is Artinian, and so the problem is reduced to the previous case, thus $A/I=B/I$. Since $I\subset B$, $A=B$, and therefore $R=T$. This completes the proof.

Although we have $T_{Z^*}=R_{Z^*}$, we do not necessarily have $R=T$. Let $T=Z[x]$ and let R be the subring generated by $\{1, 2x, x^2, x^3, \ldots\}$. Then $t^2 \in R$ for every $t \in T$, but $R \neq T$.

We conclude this paper with an example of a prime ring T, without unity, radical over a subring R which is not prime, where char. $T=0$. Let F be the free Z-algebra on countably many noncommuting variables, $x_1, x_2, \ldots$. We assume that Z is not embedded in F. Since F is countable, we can order the elements, $f_1, f_2, \ldots$. Let S be the set of monomials occurring as terms in the set $\{f_k^k : k=1, 2, \ldots\}$, and let S' be the multiplicative closure of S. Let E be the subring of F generated by S, and let I be the two-sided ideal generated by $\{x_1 s x_1 : s \in S'\}$. Set $T=F/I$ and $R=E/E \cap I$. That T is radical over R, and that R is not semiprime follows easily from our construction. Let m_1 and m_2 be nonzero monomials in T, and let h be an integer such that x_h does not occur in any generator g of I with (degree g)$\leqq$(degree m_1+degree m_2+1). Then $m_1 x_h m_2 \notin I$, hence $m_1 x_h m_2 \neq 0$. It quickly follows that T is prime.

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A RESULT ON SUMS OF SQUARES

BY

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In this note we give an elementary proof of the following.

THEOREM 1. *Let* $n \geq 1$ *be an integer. Then, every positive even integer less than or equal to* $n(n^2-1)/3$ *can be expressed as a sum of* n *squares of integers from the set* $\{0, 1, 2, \ldots, n-1\}$.

Theorem 1 follows from Lagrange's Four Square Theorem. Indeed, using Lagrange's Theorem we can show that $[n/3]+5$ squares are sufficient (a number smaller than n, for $n>7$). This will be proved in Theorem 2. The virtue of Theorem 1, is that the proof is completely elementary, requiring no Number Theory and moreover gives a constructive method for finding such a representation.

Theorem 1 is obtained from some remarks on permutation groups. Let S_n denote the permutation group on $\{1, 2, \ldots, n\}$. If $\sigma \in S_n$, let $m(\sigma)=\sum_{i=1}^n |\sigma(i)-i|^2$. Note that if i is the identity permutation, then $m(\iota)=0$ and if ρ is the reverse permutation, given by $\rho(i)=n+1-i$, then $m(\rho)=n(n^2-1)/3$.

PROPOSITION 1. *For* $\sigma \in S_n$, $m(\sigma)$ *is even and lies in the interval* $[0, n(n^2-1)/3]$.

Proof. We show in fact that

$$m(\sigma)+m(\rho \circ \sigma) = \frac{n(n^2-1)}{3} \tag{1}$$

Expanding we obtain

$$m(\sigma) = \sum_{i=1}^n \sigma(i)^2 + \sum_{i=1}^n i^2 - 2\sum_{i=1}^n i\sigma(i)$$

Thus

$$m(\sigma) = 2\sum_{i=1}^n i^2 - 2\sum_{i=1}^n i\sigma(i), \tag{2}$$

which shows that $m(\sigma)$ is even,
Similarly,

$$m(p \circ \sigma) = 2\sum_{i=1}^n i^2 - 2\sum_{i=1}^n i(n+1-\sigma(i)) \tag{3}$$

From the fact that $\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$, we find that adding (2) and (3) gives us (1).

PROPOSITION 2. *Let* $n \geq 4$, *and let* w *be an even integer between* 0 *and* $n(n^2-1)/3$. *Then, there exists a* $\sigma \in S_n$ *with* $m(\sigma)=w$.

Proof. The proof is by induction. For $n=4$, the result is true by inspection. So let $n>4$. From equation (1), we can assume that $w \leq n(n^2-1)/6$. Since $n(n^2-1)/6 \leq (n-2)(n-1)(n)/3$ for $n \geq 5$, it follows that $w \leq (n-2)(n-1)(n)/3$. So by the inductive hypothesis, there exists $\hat{\sigma} \in S_{n-1}$ such that $m(\hat{\sigma})=w$. We let $\sigma(i)=\hat{\sigma}(i)$, $1 \leq i \leq n-1$ and $\sigma(n)=n$ and thus $m(\sigma)=w$.

Proof of Theorem 1. If $n=1$, 2 or 3 the result is true by inspection. If $n \geq 4$, the result follows from Proposition 2 and the fact that $|\sigma(i)-i| \in \{0, 1, \ldots, n-1\}$.

REMARK. This proof gives us a constructive method for finding an expression for the integer w as a sum of squares. For example, if $w=62$, $n=6$. The problem is to solve $m(\sigma)=62$, $\sigma \in S_6$.

For $n=6$, $n(n^2-1)/3=70$ so, we have $m(\rho \circ \sigma)=8$

From S_4 we see that if $\hat{\sigma}=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 3 & 2\end{pmatrix}$ then $m(\hat{\sigma})=8$.

So $\rho \circ \sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 4 & 3 & 2 & 5 & 6\end{pmatrix}$ and hence

$\sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 6 & 3 & 4 & 5 & 2 & 1\end{pmatrix}$

i.e. $62=5^2+1^2+1^2+1^2+3^2+5^2$

THEOREM 2. *Let* $n \geq 1$. *Then every positive integer not greater than* $n(n^2-1)/3$ *can be expressed as a sum of* $[n/3]+5$ *squares of integers from the set* $\{0, 1, 2, \ldots, n-1\}$.

Proof. If $n=1$ or 2, the proof is trivial. Let $n \geq 3$, and $1 \leq w \leq n(n^2-1)/3$. Then $w=k(n-1)^2+\ell$, where $0 \leq \ell < (n-1)^2$. By Lagrange's Theorem [1], ℓ is a sum of 4 squares of integers from $\{0, 1, 2, \ldots, n-2\}$.
Now

$$k=\left[\frac{w}{(n-1)^2}\right] \leq \left[\frac{n(n^2-1)}{3(n-1)^2}\right]=\left[\frac{n(n+1)}{3(n-1)}\right]$$

But $n(n+1)/3(n-1) \leq (n/3)+1$ so it follows that $k \leq [n/3]+1$. Thus, the number of squares required is at most $[n/3]+5$, which concludes the proof.

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AN EQUATIONAL SPECTRUM GIVING CARDINALITIES OF ENDOMORPHISM MONOIDS[1]

BY

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ABSTRACT. By determining the spectrum of a particular set of equations of type $\langle 2, 2, 0, 0, 0\rangle$ it is shown that a positive integer n is the cardinality of the endomorphism monoid of a universal algebra of the form $\mathfrak{A}\times\mathfrak{A}$ if and only if n is square.

It was shown in [2] that for any universal algebra $\mathfrak{U}$ the cardinality of $\mathrm{End}(\mathfrak{U}\times\mathfrak{U})$ is square. Conversely, assuming the axiom of choice in the guise of the assertion that every infinite cardinal is its own square, one can readily deduce from the construction in Theorem 2.2 of [2] that every infinite cardinal is the power of the endomorphism monoid of $\mathfrak{U}\times\mathfrak{U}$ for a suitably chosen multi-unary algebra $\mathfrak{U}$. The goal of the present note is to establish this fact for finite non-zero squares as well.

By Theorem 1.3 of [2], the problem is equivalent to showing that the set of non-zero finite squares is contained in (hence equal to) the spectrum (i.e., the set of cardinalities of finite models) of the following set Σ of equations in two binary operation symbols, denoted by $*$ and juxtaposition, and three nullary operation symbols 1, d_0, and d_1.

$$\Sigma:\begin{cases} x(yz) = (xy)z \\ x1 = 1x = x \\ d_i d_j = d_i \qquad (i, j \in \{0, 1\}) \\ (x\, d_0) * (x\, d_1) = x \\ (x * y)\, d_0 = x\, d_0 \\ (x * y)\, d_1 = y\, d_1 \end{cases}$$

THEOREM. *For every positive integer n there is a multi-unary algebra $\mathfrak{U}$ such that* $|\mathrm{End}(\mathfrak{U}\times\mathfrak{U})| = n^2$.

Proof. As the theorem is trivial for $n=1$, assume $n>1$ and set $k=[n/2]$. Let M be any monoid of cardinality $2k$ containing an element t such that $t\neq t^2=e$ (the identity element) and there is a retraction ψ of M onto $\{e, t\}$. (E.g., take M to be the direct product of any k-element monoid with the two-element group.) If n is odd let N denote the monoid obtained from M by adjoining a new element 0 and extending multiplication in the usual way, $x0=0x=0$ for all $x \in N$. If n is even set $N=M$; in either case $|N|=n$. Let e' and t' respectively denote $\{e\}\psi^{-1}$ and $\{t\}\psi^{-1}$ if n is even, but if n is odd set $e'=\{e\}\psi^{-1} \cup \{0\}$ and $t'=\{t\}\psi^{-1} \cup \{0\}$.

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Define on $N \times N$ an algebraic system of type $\langle 2, 2, 0, 0, 0\rangle$ as follows. Set $1=\langle e, e\rangle$, $d_0=\langle e, t\rangle$, $d_1=\langle t, e\rangle$. Letting x_0 and x_1 denote respectively the left and right components of an element x of $N \times N$, define a binary operation $*$ by setting $x * y=\langle x_0, y_1\rangle$ for all $x, y \in N \times N$. To define the remaining binary operation, first note that $N \times N=A \cup B \cup C \cup D$, where $A=e' \times e'$, $B=e' \times t'$, $C=t' \times e'$, and $D=t' \times t'$. Now define multiplication by stipulating that for all $x, y \in N \times N$,

$$xy = \begin{cases} \langle x_0y_0, x_1y_1\rangle & \text{if} \quad y \in A, \\ \langle x_0y_0, x_0y_1\rangle & \text{if} \quad y \in B, \\ \langle x_1y_0, x_1y_1\rangle & \text{if} \quad y \in C, \\ \langle x_1y_0, x_0y_1\rangle & \text{if} \quad y \in D. \end{cases}$$

(Note that if n is odd it is necessary to observe that this multiplication is well-defined.)

Since ψ is identity on $\{e, t\}$ we have $1 \in A$, $d_0 \in B$, and $d_1 \in C$. Verification that the equations in Σ are identities in the structure just defined is rather immediate except for the first equation, associativity of multiplication, whose verification requires sixteen cases, arising from the respective assignment of y and z to A, B, C, D. However, each of the cases is very easily checked once one knows that whenever $Y, Z \in \{A, B, C, D\}$ there is a unique $W \in \{A, B, C, D\}$ such that $yz \in W$ for all $y \in Y$ and $z \in Z$. Using the fact that ψ is an endomorphism of M it is easily shown that such a W exists and is given as the intersection of the Y-row and Z-column in the table

	A	B	C	D
A	A	B	C	D
B	B	B	B	B
C	C	C	C	C
D	D	C	B	A.

Thus Σ has a model of power n^2, and the proof is concluded.

The fact that every model of Σ (hence every monoid of the form $\mathrm{End}(\mathfrak{A} \times \mathfrak{A})$ for a universal algebra $\mathfrak{A}$) has square cardinality was shown in [2] by observing that in any model M of Σ the map $x \to \langle xd_0, xd_1\rangle$ is a bijection of M onto $Md_0 \times Md_0$. An alternative proof can be obtained by noting that any non-trivial model of Σ is, with respect to $*$, a rectangular band admitting an anti-automorphism of order two, namely the map $x \to x(d_1 * d_0)$; a result of Evans [1] asserts that the cardinality of such a band is square.

Finally, we remark that the construction used in proving the theorem of this note bears some similarity to the proof of Theorem 3.1 of [2]. Moreover it can be shown that in the case where n is even the present result follows from the construction in Theorem 3.1.

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ON COUNTABLY, σ-, AND SEQUENTIALLY BARRELLED SPACES

BY

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The first author [2] calls a Hausdorff locally convex (abbreviated to l.c.) space (E, u) countably barrelled if each $\sigma(E', E)$-bounded subset of E', which is the countable union of equicontinuous subsets of E', is itself equicontinuous.

De Wilde and Houet [1] call a l.c. space (E, u) σ-barralled (which is the same as ω-barrelled of [5]) if each $\sigma(E', E)$-bounded sequence in E' is equicontinuous.

Webb [7] calls a l.c. space (E, u) sequentially barrelled if each $\sigma(E', E)$-convergent sequence in E' is equicontinuous.

Every barrelled space is countably barrelled; every countably barrelled space is σ-barrelled; and every σ-barrelled space is sequentially barrelled. Iyahen [3] and Morris and Wulbert [6] have given examples of countably barrelled spaces which are not barrelled. In this note we give two examples of sequentially barrelled spaces which are not σ-barrelled.

Example 1. It is known that the space l^∞ of all bounded sequences (real or complex) is a perfect sequence space and l^1 is its Köthe dual ([4], page 406). Consider $(l^\infty, \tau(l^\infty, l^1))$, where $\tau(l^\infty, l^1)$ is the Mackey topology on l^∞. Then, $(l^\infty, \tau(l^\infty, l^1))$ is sequentially barrelled ([7], page 354) and we show, that it is not σ-barrelled. Suppose it is σ-barrelled. Then, being separable ([7], page 357), it is barrelled ([1], page 260). But it is known that it is not barrelled ([7], page 357), which is a contradiction.

Remark. That $(l^\infty, \tau(l^\infty, l^1))$ is not σ-barrelled also follows from [5], pages 100 and 102.

Example 2. The space $(l^1, \tau(l^1, c_0))$, where c_0 is the space of null sequences, is a sequentially barrelled space ([7], page 357). That it is not σ-barrelled follows from ([5], page 102).

It is not yet known whether or not every σ-barrelled space is countably barrelled.

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FORMULA FOR THE *N*th PRIME NUMBER

BY
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In this note we give a simple formula for the nth prime number. Let p_n denote the nth prime number ($p_1=2, p_2=3$, etc.). We shall show that p_n is given by the following formula.

THEOREM.

$$p_n = \sum_{i=0}^{n^2} \left(1 \dot{-} \left(\left(\sum_{j=0}^{i} r((j \dot{-} 1)!^2, j)\right) \dot{-} n\right)\right).$$

Here $r(x, y)$ denotes the remainder upon division of x by y. (We take $r(x, 0)=x$). Proper subtraction, $x \dot{-} y$, is defined as follows: If $y \leq x$ then $x \dot{-} y = x-y$, and if $x<y$ then $x \dot{-} y=0$.

Proof. If j is prime, Wilson's theorem asserts that $(j-1)! \equiv -1 \bmod j$. When j is composite, $(j-1)! \equiv 0 \bmod j$ with the single exception of $j=4$. If $j=4$ then $(j-1)! \equiv 2 \bmod j$. In any case we see that

$$r((j-1)!^2, j) = \begin{cases} 1 & \text{if } j \text{ is prime,} \\ 0 & \text{if } j \text{ is composite.} \end{cases} \tag{2}$$

It follows from (2) that, if $\pi(i)$ denotes the number of primes $\leq i$ then

$$\pi(i) = \sum_{j=1}^{i} r((j-1)!^2, j), \qquad (i = 1, 2, 3, \ldots). \tag{3}$$

The function $C(a, n)=1 \dot{-} ((1+a) \dot{-} n)$ is easily seen to be the characteristic function of the relation $a<n$. That is to say, $C(a, n)=1$ for $a<n$ and 0 otherwise. Now $\pi(i)<n$ if and only if $i<p_n$. Therefore $C(\pi(i), n)=1$ for $i<p_n$ and 0 otherwise. Hence we see that

$$p_n = \sum_{i=0}^{k} C(\pi(i), n), \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

whenever k is large enough (i.e. $k \geq p_n - 1$).

Bertrand's postulate implies that $p_n \leq 2^n$. Therefore we could take $k=2^n$ in (4). However, a much more economical bound is possible. J. Barkley Rosser and Lowell Schoenfeld [3] have proven that $p_n<n(\log n+\log \log n)$, for $5<n$. Thus we may actually take $k=n^2$ in (4).

Now for $j=0$, $r((j \dot{-} 1)!^2, j)=1$. Hence (3) implies that

$$1+\pi(i) = \sum_{j=0}^{i} r((j \dot{-} 1)!^2, j), \qquad (i = 0, 1, 2, \ldots). \tag{5}$$

To obtain the formula of the theorem we need only substitute (5) into (4).

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CARACTERISATION DES ESPACES PARFAITS

PAR
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Dans cette note, nous employons la terminologie et les définitions de [1] et [3].

A un sous-espace vectoriel W d'un espace vectoriel V pré-ordonné par un cône convexe pointé V^+ avec élément unité e, nous associons la semi-norme suivante: pour $x \in V$, $p_W(x)=\inf\{\varepsilon>0 \mid \exists \omega_\varepsilon \in W \text{ tel que } -\omega_\varepsilon-\varepsilon e\leq x\leq \omega_\varepsilon+\varepsilon e\}$. Posons $J(W)=\{x \mid p_W(x)=0\}=\{x \in V \mid \forall \varepsilon>0, \exists \omega_\varepsilon \in W, -\omega_\varepsilon-\varepsilon e\leq x\leq \omega_\varepsilon+\varepsilon e\}$. $J(W)$ est un idéal. Nous dirons qu'on sous-espace W d'un espace vectoriel pré-ordonné V avec élément unité e est parfait si pour tout $x \in W$, pour tout $\varepsilon>0$, il existe $\omega_\varepsilon \in W$ tel que $-\omega_\varepsilon-\varepsilon e\leq x\leq \omega_\varepsilon+\varepsilon e$. Clairement W est un espace parfait $\Leftrightarrow W\subseteq J(W)$.

THEOREME 1. *Si W est un sous-espace vectoriel de l'espace vectoriel ordonné V avec élément unité e, alors $W\subseteq J(W) \Leftrightarrow W^\perp$ est un idéal dans V'.*

Preuve. ($\Rightarrow$) Soit $f \in W^\perp$ et $-f\leq g\leq f$, alors $f+g\geq 0$ et $f-g\geq 0$. Pour $x \in W$, on a $\forall \varepsilon>0$, $\exists \omega_\varepsilon \in W$ tel que $-\omega_\varepsilon-\varepsilon e\leq x\leq \omega_\varepsilon+\varepsilon e$. Si on évalue $f+g$ et $f-g$ en ces points, on obtient les inégalités

$$-\varepsilon(f(e)+g(e))-g(\omega_\varepsilon) \leq g(x) \leq \varepsilon(f(e)+g(e))+g(\omega_\varepsilon) \text{ et}$$
$$-\varepsilon(f(e)-g(e))+g(\omega_\varepsilon) \leq -g(x) \leq -\varepsilon(f(e)-g(e))-g(\omega_\varepsilon)$$

d'où l'on tire $-\varepsilon f(e)\leq g(x)\leq \varepsilon f(e)$, ε étant arbitraire, on conclut que $g(x)=0$, c'est-à-dire $g \in W^\perp$.

($\Leftarrow$) Puisque $W\nsubseteq J(W)$, on a $p_W(x)\not\equiv 0$ sur W. Ainsi, il existe $\omega \in W$ tel que $p_W(\omega)\neq 0$. Par le théorème de Hahn-Banach, il existe une fonctionnelle linéaire $\phi: W\to R$ telle que pour tout $x \in W$, $|\phi(x)|\leq p_W(x)$ et $\phi\not\equiv 0$ sur W. De plus, pour $x \in [-e, e]$ un ensemble radial et $x \in W$ on a $-(-x+2e)\leq x\leq -x+2e$ d'où $p_W(x)\leq 2$. On peut donc prolonger les fonctionnelles linéaires ϕ et $-\phi$ à deux fonctionnelles linéaires positives f et g définies sur V. (Peressini [4], p. 82). Or, pour tout $x \in W$, $f(x)+g(x)=\phi(x)-\phi(x)=0$. Donc $f+g \in W^\perp$. De plus, il est clair que $f \notin W^\perp$ et $g \notin W^\perp$ et $-(f+g)\leq f\leq f+g$. Donc $W^\perp$ n'est pas un idéal dans V'.

COROLLAIRE 2. (*Théorème de Ellis* [3]). *Un idéal I dans un espace vectoriel ordonné V avec élément unité e est parfait si et seulement si $I^\perp$ est un idéal dans V'.*

Le lemme suivant nous permettra d'obtenir la caractérisation de Bonsall pour les idéaux maximaux parfaits comme corollaire du théorème 1.

LEMME 3. *Dans un espace vectoriel pré-ordonné V, si W est le noyau d'une* *fonctionnelle positive non nulle f, alors $W^\perp$ est un ideal$\Leftrightarrow f$ est sur une génératrice extrémale.*

Preuve. ($\Rightarrow$) Il est clair que $f \in W^\perp$. Supposons que $f=f_1+f_2$, $f_1 \geq 0$, $f_2 \geq 0$, alors $-f \leq f_1 \leq f$, $-f \leq f_2 \leq f$, d'où $f_1, f_2 \in W^\perp$. Ainsi, $f=f_1=f_2$ sur W et alors $f_1=\lambda_1 f$, $f_2=\lambda_2 f$, $\lambda_1>0$, $\lambda_2>0$. Donc f est extrémale.

($\Leftarrow$) Supposons f extrémale et considérons $g \in W$, $g \not\equiv 0$. Dans ce cas puisque W est de codim 1, f et g ont le même noyau et ainsi $g=\lambda f$, $\lambda>0$. Puisqu'il en est ainsi, il suffit de vérifier que $-f \leq h \leq f \Rightarrow h \in W^\perp$. Or, si $-f \leq h \leq f$, on a $f+h \geq 0$ et $f-h \geq 0$ et $f=(f+h)/2+(f-h)/2$. Puisque f est extrémale, on en conclut que $h=\alpha f$, $\alpha>0$ et alors $h \in W^\perp$.

COROLLAIRE 4. (*Théorème de Bonsall* [1]). *Dans un espace vectoriel ordonné (avec unité), un idéal maximal W est parfait si et seulement si toute fonctionnelle positive f associée* à *W appartient* à *une génératrice extrémale.*

Preuve. En vertu du lemme 3, on a f est extrémale $\Leftrightarrow W^\perp$ est un idéal et par le théorème 1, $W^\perp$ est un idéal $\Leftrightarrow W \subseteq J(W)$ d'où f est extrémale $\Leftrightarrow W$ est un idéal maximal parfait.

Je tiens à exprimer mes remerciements à M. Serge Dubuc pour ses nombreuses suggestions et ses commentaires judicieux.

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COMMENTS ON THE SPINOR STRUCTURE OF SPACE-TIME[*]

BY

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ABSTRACT. A simpler proof of the theorem on the spinor structure of space-time is given. Some geometrical insights are provided.

The definition and the implications of the existence of a spinor structure on a space-time[*] have been elucidated by Geroch [1]. By construction, he has established that *a necessary and sufficient condition for a non-compact space-time M to admit a spinor structure is that there exists on M a global system of orthonormal tetrads* [1]. The same is true for a compact space-time M [2]. This short note provides a simpler proof of the theorem and some geometrical insights into the theorem are given.

It is well-known that the space-time M possesses a spinor structure if and only if the second Stiefel-Whitney class of M, $w_2(M)$, vanishes [3], where the q-th Stiefel-Whitney class is a characteristic cohomology class of $H^q(M, Z_2)$. The geometrical meaning for the condition $w_q(M)=0$, for $q=1, 2, 3, 4$, is equivalent to the existence of a continuous field of orthogonal $(4-(q-1))$-frames over the q-dimensional skeleton of M [5]. Consequently, saying that the space-time M admits a spinor structure is equivalent to claiming that an orthogonal triad can be placed at each point of every two-surface of M in a continuous way. To say that there exists on the space-time M a global system of orthogonal tetrads is tantamount to claiming that $w_q(M)=0$ for all q, $q=1, 2, 3, 4$. It is then obvious that we only have to prove that if the space-time M admits a spinor structure, then there exists on M a global system of orthonormal tetrads.

The q-th Stiefel-Whitney class of M is equal to the obstruction of the vector bundle ξ over M with Z_2 coefficients [4]. But the obstruction $\theta_q(\xi)$, is a characteristic cohomology class of $H^q(M, \pi_{q-1}(F))$. For the bundle of orthonormal frames on M, the structure group is the proper Lorentz group, L_0, which is homeomorphic to $P^3 \times R^3$ where P^3 is the 3-dimensional real projective space. Since the space-time M is orientable, we have $w_1(M)=0$. $w_2(M)=0$ because the space-time admits a spinor structure. $\theta_3(\xi) \in H^3(M, \pi_2(L_0))$, but $\pi_2(L_0)=0$, thus $\theta_3(\xi)=0$. Consequently $w_3(M)=0$. For non-compact space-time M, $H^4(M)=0$, thus $w_4(M)=0$.

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[*] A space-time M is an orientable 4-dim differentiable manifold with a metric of Lorentz signature $(-, +, +, +)$.

If M is a compact space-time, the Euler-Poincaré characteristic class of M vanishes [5, p. 203], thus $w_4(M)=0$. The proof is completed.

It is evident from the proof that the theorem is not true if the dimension of the space-time is greater than four.

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ON THE DERIVATIVES OF THE Γ-FUNCTION

BY
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1. The coefficients of the two series

$$
\begin{aligned}
\Gamma(1+z) &= \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < 1, \\
1/\Gamma(1+z) &= \sum_{n=0}^{\infty} g_n z^n, \qquad |z| < \infty,
\end{aligned} \tag{1}
$$

are recursively given by Nielsen [1]: $c_0=g_0=1$ and

$$
\begin{aligned}
(n+1)c_{n+1} &= \sum_{j=0}^{n}(-1)^{j+1} s_{j+1} c_{n-j}, \\
(n+1)g_{n+1} &= \sum_{j=0}^{n}(-1)^{j} s_{j+1} g_{n-j},
\end{aligned}
$$

where s_1 is the Euler constant γ and for $n>1$ $s_n=\zeta(n)$. We have then from (1): $c_n=(n!)^{-1}\Gamma^{(n)}(1)$. Nielsen (loc. cit., p. 40) remarks that no simple direct representation of the coefficients c_n is known. Using the Faa di Bruno formula for the n-th derivative of a compound function, he shows, following Schlömilch, that

$$
c_n = (-1)^n \sum_{k=1}^{n} \frac{1}{k!} \sum \prod_{i=1}^{k} s_{r_i}/r_i
$$

where the second summation is over all positive solutions of $r_1+\cdots+r_k=n$ and the s_1 are as above. In this note we use a completely self-contained elementary method to calculate the coefficients c_n or, what amounts to the same thing, the derivatives of the Γ-function. We show that

$$
\begin{aligned}
\Gamma^{(k)}(1) = 2\lim_{n\to\infty} & \\
& \left[(-1)^n \frac{n^{n+2}}{(n+1)!} \sum_{j=0}^{k/2} (2j)!\, \zeta(2j)(1-2^{1-2j}) \binom{k}{2j} \cdot \frac{d^n(\log^{k-2j} a/a)}{da^n}\bigg|_{n=a}\right].
\end{aligned} \tag{2}
$$

2. Starting with the Euler integral

$$
\Gamma(1+x) = \int_0^{\infty} e^{-t} t^x \, dt
$$

we have by differentiating k times and putting $x=0$

$$
\Gamma^{(k)}(1) = \int_0^{\infty} e^{-t} \log^k t \, dt.
$$

To evaluate this we begin by setting

$$I_k(a) = \int_0^\infty \frac{\log^k x}{(a+x)^2}\,dx, \qquad k = 0, 1, \ldots. \tag{3}$$

The reason for this indirection is following. Once $I_k(a)$ is known we have, differentiating n times with respect to a and setting $a=n$,

$$\int_0^\infty \frac{n^{n+2}}{(n+x)^{n+2}} \log^k x\,dx = (-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n I_k(a)}{da^n}\bigg|_{a=n}.$$

Since

$$\lim_{n\to\infty} \frac{n^{n+2}}{(n+x)^{n+2}} = e^{-x},$$

justifying the passage to the limit on n under the integral above we find

$$\int_0^\infty e^{-x} \log^k x\,dx = \lim_{n\to\infty}\left[(-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n I_k(a)}{da^n}\bigg|_{a=n}\right].$$

3. To evaluate $I_k(a)$ we put $x=ay$ in (3) getting

$$I_k(a) = a^{-1}\sum_{j=0}^{k}\binom{k}{j} b_j \log^{k-j} a \tag{4}$$

where

$$b_j = \int_0^\infty \frac{\log^j y}{(1+y)^2}\,dy. \tag{5}$$

The integral (5) is evaluated by being broken up into two parts corresponding to the integrals $0-1$ and $1-\infty$; letting $y=1/u$ in the first one we get

$$b_j = [1+(-1)^j]\int_1^\infty \frac{\log^j y}{(1+y)^2}\,dy. \tag{6}$$

This is evaluated by the substitution $x=\exp(y)$ and observing that

$$(1+e^x)^{-2}e^x = (1+e^{-x})^{-2}e^{-x};$$

expanding in series and integrating term by term we have

$$b_j = 0 \quad \text{for } j \text{ odd}, \qquad b_j = 2(j!)(1-2^{1-j})\zeta(j) \quad \text{for } j \text{ even}. \tag{7}$$

This holds for all j including $j=0$ as can be verified, either from (6) directly or recalling that $\zeta(0)=-\frac{1}{2}$. Now, putting together (3)–(7) we get (2).

4. It may be verified that for the first few values of k (2) gives us the correct c_k's in (1). For instance, with $k=1$

$$\Gamma'(1) = \lim_{n\to\infty}\left[(-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n(\log a/a)}{d^n a}\bigg|_{a=n}\right].$$

The n-th derivative of $a^{-1}\log a$ is computed by the Leibniz rule and we get

$$\Gamma'(1) = -\lim \frac{n}{(n+1)}\left(\log n - \sum_{j=1}^{n} 1/j\right) = \gamma.$$

Similarly, we verify that

$$\Gamma''(1) = \gamma^2+\pi^2/6, \qquad \Gamma'''(1) = -\gamma^3-2\zeta(3)+\pi^2\gamma/6, \text{ etc.}$$

Here it is to be observed that the quantity $\zeta(3)$ does not arise from the b_j of (7) but from the limit of the n-th derivative of $a^{-1}\log^3 a$.

5. The foregoing allows us to evaluate some improper integrals. For instance integrating (3) with respect to a from p to q we evaluate

$$\int_0^\infty \frac{\log^k x}{(x+p)(x+q)}\,dx,\ p \neq q,\ p \text{ and } q > 0.$$

In conclusion, we mention the well-known theorem of Hoelder which states that the Γ-function satisfies no polynomial differential equation of the type

$$P(x, y, y', \ldots, y^{(n)}) = 0.$$

Thus any formulas, recursive or otherwise, for its n-th derivative are apt to be complicated.

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LA 7e CONFÉRENCE JEFFERY-WILLIAMS PRÉSENTÉE LE 7 JUIN 1974 À L'UNIVERSITÉ LAVAL À L'OCCASION DE LA 28e RÉUNION ANNUELLE DE LA SOCIÉTÉ MATHÉMATIQUE DU CANADA

PAR
HANS J. ZASSENHAUS

On the Minkowski-Hilbert Dialogue on Mathematization

Introduction. Il me fait un très grand plaisir de vous présenter aujourd'hui la conférence Jeffrey-Williams nommée ainsi en l'honneur des distingués fondateurs de la Société Mathématique du Canada.

Il y a beaucoup d'évidences indirectes d'un vif échange d'idées entre Herman Minkowski et David Hilbert, pendant la période de temps entre la soutenance de thèse de Hilbert (décembre 1884) et la mort prématurée de Minkowski (le 12 janvier 1909). Ces évidences se trouvent dans plusieurs travaux des deux grands mathématiciens, tel que le "Zahlbericht" et le discours de Hilbert à Paris sur les problèmes non résolus en mathématiques et la "Geometrie der Zahlen" et les "Diophantische Approximationen" par Minkowski.

Récemment on a publié les lettres de H. Minkowski à D. Hilbert écrites entre 1885 et 1908. Celles-ci donnent une évidence directe de l'interaction entre H. Minkowski et D. Hilbert et du rôle de A. Hurwitz qui encourageait cette interaction en intervenant amicalement.

Parmi les nombreux sujets de discussion souvent soulevés, on a porté particulièrement attention, en citant des détails contenus dans les lettres, à la relation entre Minkowski et Hilbert d'une part, et les mathématiciens contemporains français d'autre part, surtout Hermite, Jordan, et Poincaré, ainsi que Picard et Hadamard. On y remarque un intérêt soutenu pour Dedekind et Kronecker, pour la préparation de la "Geometrie der Zahlen", pour les questions des mathématiques pures et appliquées, et pour le discours bien connu de Hilbert à Paris (1900).

En accord avec C. Reid, on constate que Minkowski et Hilbert se considéraient en compétition avec les grands mathématiciens français, Poincaré en particulier. Cette attitude a motivé en grande partie la préparation et le contenu du discours à Paris. Cependant, on perçoit un fort courant d'admiration de la part de Minkowski pour les travaux de Hermite, Hadamard, C. Jordan et d'autres mathématiciens français et une satisfaction profonde à la suite de l'intérêt accordé à son propre travail par Hermite et Picard.

Il semble qu'il y ait eu de la part de Minkowski et Hilbert un certain désir de dévaloriser les travaux de Dedekind et Kronecker aux yeux de leurs contemporains allemands. En même temps, il y avait un échange d'idées entre Dedekind et Minkowski et toujours une appréciation du génie inventif de Kronecker de la part de Minkowski.

Minkowski a passé de nombreuses années à essayer de bâtir les fondements d'une théorie constructive des nombres. Les attitudes de Minkowski, Hilbert, et F. Klein en face des applications étaient complètement différentes, attitudes dont les conséquences sont encore apparentes de nos jours.

L'interprétation de Minkowski de "l'art pour l'art" dans les mathématiques etait tout-à-fait differente de celle de la premiére génération d'auteurs de Bourbaki.

There is abundant evidence of a lively and continuing discourse on basic ideas of the mathematical sciences between Hermann Minkowski (1864–1909) and David Hilbert (1862–1943), those two great mathematicians who laid the foundations for the mathematics of our century as many of us bear witness.

Not only do you see it, you can feel it in the warm afterglow of Hilbert's memorial speech on his dearest friend.(1)

You may remember from C. Reid's Hilbert biography(2) the three friends Adolf Hurwitz (1859–1919), Herman Minkowski, and David Hilbert walking 'under the apple tree' in deep discussion of basic theories of their subject.

Hurwitz, the senior of the three by a few years, initiated his two younger colleagues into what we may call now conceptual mathematics of a universalist direction. With the passing years the senior friend changed his role from an initiator to a sympathetic witness and commentator of the discussion that was going on between Minkowski and Hilbert.

What is it about?

Though there is much indirect evidence in the collected works of the three congenial friends and though we have seen and felt one of the two witnessing already, of *direct evidence* there are extant presently only the letters of Minkowski to Hilbert from 1885 to 1908, and of these mainly the correspondence from February 14, 1885 to July 28, 1900.(3)

In spite of all efforts to locate the letters of Hilbert to Minkowski their present existence and location remains uncertain. Minkowski's oldest daughter (Mrs. L. Rüdenberg) remembers having read them about 1930 and found them to be more businesslike in tone, less colorful than her father's letters. When Minkowski moved to Göttingen in 1902 the discourse was continued orally, probably often during one of their lovely walks in the woods near Göttingen; the need for elaboration of ideas in letter form had ceased more or less.

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(1) s. Minkowski [2] I, V–XXXI.

(2) s. Reid, p. 14.

(3) The letter of July 28, 1900 describes the impact of Hilbert's Paris address on Minkowski.

As anybody glancing through the letters will realize they are honest to goodness letters between friends (only one side still talking!) notwithstanding the courtesy and dignified friendliness prevailing in them which has moved one of my American colleagues to comment: "They sound as if they were written for posterity."

In fact they are a mélange of family correspondence, gossip, trivia, mathematical policy and mathematical discussion. There may be no signature at all, or as once happened, Minkowski signed with Hilbert's name.

It remains for the reader to discover ongoing lines of thought, he is *not* primed in any way.

For today's commentary I select passages referring to certain French mathematicians, to Dedekind and Kronecker, to the preparation and continuation of the "Geometrie der Zahlen", to "applied and pure mathematics", and to Hilbert's Paris address.

In the absence of a published translation of the letters, permit me to quote extensively in my English translation.

I. On French Mathematicians

February 14, 1885: I am resolved to publish some of my results already now. They will appear in the third issue of the current volume of Crelle.⁽⁴⁾

Indeed, suddenly I got afraid I could again be deprived of my joy. Poincaré of whose many-sided and swift working talents you certainly will have heard already, has started on a series of investigations not long ago which would find their brilliant completion by those very theorems of mine. It could happen indeed that he would find this completion now after the publication of my prize-essay.⁽⁵⁾

April 26, 1886 (written while Minkowski served with the Prussian army and Hilbert had gone on a postdoctoral trip to Paris): Please tell me in minutest detail what happened to you since last October. If one of the big men, Jordan or Hermite, would perhaps remember me, please give them my best recommendations, and explain to them that I am not lazy by nature.

December 29, 1887: For the New Year my best wishes. Let us hope that neither the French nor the Russians will disturb our circles.

June 2, 1893: The Poincaré prize winning memoir⁽⁶⁾ did not impress me quite as much as it did impress Nöther, even though I believe to have it understood not

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(4) s. Minkowski [3].

(5) s. Minkowski [4].

(6) In 1885 King Oscar II of Sweden established a prize of 2500 Swedish crowns and a gold medal to reward an important discovery in the field of mathematical analysis on the occasion of his 60th birthday, January 21, 1889. The prize committee (K. Weierstrass, Charles Hermite, and G. Mittag-Leffler recommended unanimously the award of the prize to H. Poincaré for his long memoir on the three body problem published in 1892 and 1893 as Vol. I, II of "Les Méthodes nouvelles de la Mécanique céleste." S. Poincaré [1].

any less. I also had to referee a second larger and fairly interesting paper(7) of Poincaré on potential theory; I must confess, I would never be able to publish papers in the state Poincaré did.

August 20, 1894: Hermite has now studied everything I sent him of my book.(8) He writes about it full of delight to the translator: Je Crois voir la terre promise, etc.

May 30, 1896: I had a very nice letter from Picard about my book.

January 22, 1896: The papers of Jordan(9) are really quite interesting. But if already Kummer's computations were disagreeable to youth in Jordan's operations, his "pour fixer les idées" will truly nauseate you.

Comments: The revealing letter of February 14, 1885 refers to the six papers on quadratic forms published by H. Poincaré (1854–1912) between 1879 and 1882(10) in which the powerful French mathematician appropriated to himself a half century of German arithmetic developments, reinventing much of the theory of lattices, ideals, algebraic number theory, quadratic forms, and creating the beginnings of class field theory and the geometry of numbers, including some of the ideas Minkowski published a little later.(11)

However, Minkowski's fear of another anticipation by Poincaré was unfounded since the latter's work in arithmetics in subsequent years was directed towards conquering the reduction theory of forms of higher degree.(12) The problems only partially solved by Poincaré, as K. Mahler remarks, are still slumbering to be reawakened in the context of a constructive theory of algebraic numbers based on the reduction theory of higher norm forms. As we shall see in III, Minkowski worked for many years on the problem generalizing to higher algebraic number fields Lagrange's application of continued fraction algorithms to the arithmetic theory of real quadratic number fields,(13) apparently never fully to his own satisfaction and apparently not aware of the fact that his rival was trying his power on the hard problems of reduction theory.

In later letters Minkowski refers to intensive studies of the work of certain French mathematicians (like C. Hermite (1822–1901), J. Hadamard (1866–1963), C. Jordan (1838–1922)) which seem to indicate that he is after all much keener to learn than he is afraid of being anticipated. He obviously cherishes to be appreciated by his colleages "in Feindesland".

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(7) s. Poincaré [2]. Minkowski's reviews of Poincaré [1], [2] are found in Jahrbuch d. Fort schritte d. Mathematik **22** (1899), 907–914 and 977–980.

(8) No copy of the letter seems to be published.

(9) s. C. Jordan (98)–(100) and (103) of chronological list of publications in Oeuvres, 4 vls. (Paris 1961–1964).

(10) s. Poincaré [3]–[8].

(11) s. Minkowski [5].

(12) s. Poincaré [9–11] and [12] p. 21.

(13) Joseph Louis Lagrange (1736–1813), Oeuvres VII, 56 & II, 606.

But some non-quoted remarks and allegations(14) seem to show that Minkowski never rid himself entirely of his awe before H. Poincaré's power of creative achievement and fear of anticipation even when the latter became seriously ill.

Yet it is equally true that Minkowski used to say at home that life is too short to waste on trivia,(15) for him its purpose was to behold the truth, to understand it well and to expound it perfectly.

II. On R. Dedekind and Kronecker

December 20, 1893: As I had planned, I have reproduced the comment you made in Cranz(16) in an appendix to my book,(17) in part I communicated it to Dedekind(18) on the occasion of receiving a copy of the new edition of his "Zahlentheone". It is a pity, I have not rushed more to finish my book. Dedekind certainly would have needed to refer to it at several places.

April 16, 1895: The latest paper of Dedekind(19) gives me the impression that you found a fellow sufferer in Hurwitz(20) in relation to Dedekind who really seems to have adopted Kronecker's(21) ways, to have had everything already.

March 31, 1896: I like your report(22) in its austere and yet complete form extremely well, and it will certainly meet with great approval and push very much into the background the works of Dedekind and Kronecker.

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(14) Minkowski [1], 155, [2] II, 401; note that the inspiring lecture of September 21, 1908 on "Raum und Zeit" ([2] II, 431–444) makes no mention of Poincaré's contribution in establishing the basic role of the group now named after him.

(15) Minkowski [2] I, XXIX 2nd paragraph. On Poincaré and Hilbert see also Reid, p. 22–24, 62–63, 99, 106, 120–121, 125, 133, 141, 185, 186.

In an obituary published in a Bonn newspaper by Professor E. Study the impact of Minkowski': personality was summarized as follows: A man of unpretentious behavior, he hardly ever came in public view. His inner eye was focused always on the things of the mind, never on the assertion of his personality. Yet he was not without ambition—the ambition of the kind that is becoming to the research scientist: only to present perfect work to the public. Thus he published relatively few papers, not influenced by the fashions of the day. Only a later period will be able to do full justice to his import. He did not have much influence as a teacher. It appears that he did not find it easy to adjust the exposition of his ideas to the average person's ability to understand. On the whole a man of quiet disposition, Minkowski liked to let things take care of themselves. But people near him learned to appreciate his harmonious nature, so amiable with all his critical talents and full of a happy sense of humor which sometimes could become sarcastic for good reasons. He offered the example of a self-disciplined person of easy-going manners to an epoch whose ideal seems to be the development of totally different character traits.

(16) Cranz, a summer resort at the Baltic Lake near Königsberg where Hilbert was professor at the University.

(17) There is no appendix to the "Geometrie der Zahlen." Probably Minkowski refers to the appendix to the first chapter (Anhang über lineare Ungleichungen, p. 39–45).

(18) s. Dirichlet-Dedekind. The preface was completed at Bad Harzburg, September 30, 1893 and contains no reference to H. Minkowski's break-through in the geometry of numbers though there is a footnote referring to it in Supplement XI (s. Dedekind [1] III, 142).

(19) s. Dedekind [3].

(20) s. Dedekind [2].

(21) s. Frobenius.

(22) s. Hilbert [2]. The "theorem on abelian fields" (§100) is Kronecker's theorem that finite normal extensions of the rational number field with abelian automorphism group always are contained in a suitable cyclotomic field extension of the rational number field. Its proof is given in §101–104.

July 27, 1896 (referring to the Zahlbericht): Your proof of the theorem on the abelian fields(22) is really extremely simple and beautiful. In order that the reader may enjoy it fully I would like to recommend you to finish off the necessary lemmas on abelian groups with hints of the proofs in advance of §100, at least you should give a reference to the reader where he would find a full exposition of the theory of abelian groups for this purpose.(23)

April 13, 1898: Your communications about your new reciprocity law(24) made me extremely excited, and I congratulate you for the far-reaching results which you have obtained. In the interest of the progress of science, I would like to ask you possibly not to keep back the most beautiful things for yourself for a long period, as it almost seems to be the case according to your pronouncements, but disclose everything you have achieved to others so that not again the state of affairs may develop as it did after the first publications of Kronecker.(21)

Comments: April 16, 1895: In his paper "Zur Theorie der Ideale" (Göttinger Nachrichten, 1894, Werke II, p. 272–277) Dedekind refers to the recently published paper of Hilbert (Grundzüge einer Theorie des Galoisschen Zahlkörpers, Göttinger Nachrichten 1894, p. 224–236) on the theory of decomposition and ramification of prime ideals of an algebraic number field in a finite extension. He points out firstly that his earlier publications contains those laws in outline, secondly that he had communicated the full theory to Frobenius in a letter of June 8, 1882 and thirdly that the topic is capable of much further generalization.

March 31, 1896: Hilbert did succeed in pushing back the influence of Dedekind and Kronecker in number theoretical research on his German contemporaries for a generation—at a price. As two of the editors of Dedekind's collected works (E. Noether and O. Ore) wrote at the occasion of Dedekind's 100th birthday:(25) "Es ist ein Zeichen, wie Dedekind seiner Zeit voraus war, dass seine Werke noch noch heute lebendig sind, ja dass sie vielleicht erst heute ganz lebendig sind." Dedekind's deep investigations on the foundations of ideal, ring module, and lattice theories make him the father of modern algebra.

In the sense in which C. Chevalley describes modern algebra as the language of modern mathematics(26) we can see in Dedekind more than in any other single man or woman the founder of the conceptual method of mathematical theorization man or woman the founder of the conceptual method of mathematical theorization in our century. The new generation of mathematicians under the leadership of E. Noether, H. Hasse, R. Brauer, A. A. Albert, O. Schreier, E. Artin, W. Krull, and B. L. van der Waerden after the First World War realized in full detail Dedekind's self confessed desire for conceptual clarity not only in the foundations of number theory, ring theory and algebra, but on a much broader front, in all mathematical disciplines.

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(23) cp. Minkowski [1], 19–21.

(24) s. Hilbert [3].

(25) s. Dedekind [1] III, Nachwort der Herausgeber. Dedekind was born on October 6, 1831.

(26) s. Chevalley, Preface.

It is not possible to blame this situation entirely on Hilbert himself who gave a full accounting of the available literature on algebraic number theory in his 'Zahlbericht' and who himself in his earlier work on the theory of invariants proved to be Dedekind's most powerful pupil in the abstract direction of algebraic research by giving a purely existential argument for the finite generation of the ring of invariants of any system of forms.(27) As well, Hilbert in his own work was fully aware of the strength and necessity of the constructive approach which he provided for invariant theory in another less well known paper.(28) Furthermore, Hilbert in a series of classical papers inaugurated and partially anticipated the research on class field theory which in the twenties and thirties of this century almost became synonymous with higher number theory(29) so that indeed a whole generation of scholars felt forced to think Hilbert's way.(30)

Now when the force of Hilbert's influence is about spent we realize more clearly the greater depth of the theories of Hilbert's teacher Kronecker in regard to his 'liebester Jugendtraum'. See for instance the comments made by Hassell on a letter of Kronecker to Dedekind in Werke V, p. 510–515; Minkowski was one of the editors of Kronecker's collected papers!. Consider also Dedekind's broad approach which led him to the following prophetic statement:

(From a letter to Frobenius, February 5, 1883 (Werke 2, p. 419)):

> I believe firmly, there will be a time when one will find very general laws permitting to derive immediately the prime ideals of a field from its discriminant and its other invariants (which maybe also ideals of related fields); yet we may be still far removed from this goal. In the last years, I have occupied myself very little with these questions to which I am very desirous to return since they seem to be the most interesting of all....

Dedekind knew the corresponding conjectures and results of classfield theory quite well(31) as did his contemporary H. Weber(32) but Dedekind outlines a research program which does full justice to the complexity of algebraic number fields as we meet them in nature while Hilbert picks one particular case, though perhaps the only one where to this day an elegant theory is developed.

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(27) s. Hilbert [4], [5].

(28) s. Hilbert [6].

(29) s. Hilbert [3], [7], [8].

(30) "You have made us all think only that which you would have us think" Reid, 214.

(31) s. Dedekind [4].

(32) s. Weber II §208–223, III §150–169.

III. On the Geometry of Numbers

November 6, 1889: I have made quite a bit of progress on the theory of positive definite quadratic forms indeed a lot turns out to be different for forms with a larger number of variables. Perhaps you or Hurwitz will be interested in the following theorem (that I can prove on half a page):

To every positive definite quadratic form of determinant D *of* n $(\geqq 2)$ *variables one can always give the argument such integral values that the form becomes* $< nD^{1/n}$.

Hermite(33) has here for the coefficient n only $(4/3)^{1/2(n-1)}$, which is much larger in general.

December 22, 1890: I have generalized my proof for the theorem on the minimum of a positive quadratic form in an extraordinary way, I have realized that the special advantage of quadratic forms is illusory, in as much as other definite forms (though not necessarily rational forms) permit much further reaching consequences Thus I found the following result,(34) which could not be obtained by the use of quadratic forms: The discriminant of any algebraic number field which arises from an integral equation with $n-2\beta$ real and 2β complex roots is absolutely greater than

$$\left(\frac{\pi}{4}\right)^{2\beta}\frac{n^{2n}}{(n!)^2}$$

The letter of June 11, 1891 announces in detail four fundamental results of the new science of the geometry of numbers: asymptotic class number formula for positive definite quadratic forms with integral coefficients, Minkowski (Siegel-Hlawka) theorem,(35) estimates for densest lattice like sphere packing in n dimensions from above and below.(36)

August 30, 1892: With my book I have so far advanced that I shall turn to a publisher one of these days. I did not want to do it before everything was clear and definite. I have expounded all principles that I use, e.g. the auxiliary results from the theory of functions.

Another reference to the book is found in the letter of February 23, 1893 and again in the letter of June 2, 1893: Since I voluntarily contracted for awhile to paint doors and to varnish windows I must finish the job now. I hope it does not last much longer. The last item I put into my manuscript was a proof for the periodic-expansion of quadratic irrationalities as continued fractions. It is really strange, in how bad a state this old problem still was and how one based everything on a few formulae which had gotten in by change. Moreover I believe that every civilized mathematician must have a horror of the chapter on continued fractions in the textbooks. In my presentation there is hardly a trace left of computations and I believe I have suceeded to formulate more precisely the often uttered vague desire for generalization of the continued fraction theory.

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(33) s. Hermite [2].

(34) s. Minkowski [6], [7].

(35) Regarding the asymptotic class number formula, s. Minkowski [10]. Minkowski [8], [9] conjectured that any pointset of the n-dimensional space that is starred at the origin and of volume $\zeta(n)^n$ can be transformed by a volume preserving linear transformation so as to avoid all integral lattice points other than the origin. This conjecture was proved only for solid n-spheres by Minkowski [10], it was proved in general by E. Hlawka and put by C. L. Siegel into the context intended by Minkowski [9] (p. 279: Der Nachweis dieses Satzes erfordert eine arithmetische Theorie der kontinuierlichen Gruppe aus allen linearen Transformationen).

(36) s. Minkowski [10].

February 8, 1894: There can be no argument about the educational value of (classical) geometry though there is more life in other areas of mathematics.

August 20, 1894: I am about ready with continued fractions for two real quantities. The now finished version of this investigation will be, I believe, quite instructive; it has much similarity with Jacobi's algorithm;(37) my algorithm is somewhat more sophisticated, which is compensated for by the possibility of deciding more plausible questions, to which the answer for the other algorithm remains in doubt.

February 10, 1896: The complete presentation of my investigations on continued fractions has needed almost 100 pages eventually. And yet it still lacked the only satisfactory end, a characteristic criterion for cubic irrationalities which was vaguely stirring in my mind. Without this completion, which I believed to be quite near, that chapter of my book would have been unsatisfactory at least for me, . . . as now recently again Weber urged me on to publish at least what was there already I resolved to do it . . . .

March 11, 1897: For my theorem that one can make n real linear forms of determinant $\pm D$ simultaneously absolutely less than $D^{1/n}$ by integral values of the variables Hurwitz has found a ridiculously simple proof after a recent conversation so that I must be very much ashamed not to find it myself. This proof takes care also of the boundary case regarding the equality sign in $\leqq D^{1/n}$, even though there are difficulties left in that latter case. The proof is as follows. It suffices to deal with the theorem for forms with rational integral coefficients: Let the forms be

$$y_h = a_{h1}x_1 + a_{h2}x_2 + \cdots + a_{hn}x_n \qquad (h = 1, 2, \ldots, n)$$

and abs $|a_{hk}| = D$. Then $Dx_k = A_{1k}y_1 + A_{2k}y_2 + \cdots + A_{nk}y_n = \xi_k$ when the A_{hk} are rational integers ($k=1, \ldots, n$). Now there are among the systems $\xi_1, \ldots, \xi_n$ modulo D all told D systems $\xi_1, \ldots, \xi_n$ that are not congruent as is shown by unimodular reduction of $|a_{hk}|$ to a quadratic system with non zero terms only in the diagonal. Suppose Δ is the greatest rational integer not larger than $D^{1/n}$ so that $(\Delta+1)^n > D$ and if $y_h = 0, 1, 2, \ldots, \Delta (h=1, \ldots, n)$, then among those

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(37) s. Leon Bernstein; The Jacobi-Perron Algorithm, its theory and application, Lecture notes 207, Springer 1971.

$(\Delta+1)^n$ systems there are necessarily some two, say $y_1', y_2', \dots, y_n'$ and $y_1, y_2'', \dots, y_n''$ for which $\xi_1, \dots, \xi_n$ assume the same residue class systems modulo D. Then $y_1''-y_1'=b_1, \dots, y_n''-y_n'=b_n$ are all $\leqq D^{1/n}$, but not all zero, and there are rational integers $x_1, \dots, x_n$ for which the forms $y_1, \dots, y_n$ will be equal to $b_1, \dots, b_n$.

April 4, 1897: Hurwitz's proof for my theorem on linear forms really is essentially different from yours(38) though Dirichlet's idea finds its application in it too.

April 13, 1898: In recent times I have worked once again on generalizations of continued fraction algorithms. Thus each step requires long computations to which the results are in no comparable relation; in spite of everything I can't tear away myself from the subject. Certainly there are beautiful things hidden in the background.

July 20, 1898: For the computation of class numbers the bounds that the method of §42, p. 133 of my book gives are the best. But really the best methods to that end will be contained only in the second part of my book.(39) I myself do many new examples for my new algorithms, and I believe that much light, particularly on the theory of cubic extensions, will be shed due to the new computational tools.(40)

June 22, 1900: Since we parted from each other I worked with full vigor at my second volume. In its completed state I like some things quite well. I believe the computation with the new algorithms which I expound will find some friends, though perhaps not among the top mathematicians.

July 28, 1900: With my book I did not get as far as I wanted—I had hoped to present the final chapters to Hermite personally.(41) As Volterra indicated, probably Hermite will not be approachable at all.

Comments: Why does Minkowski see not much life in Geometry though he worked all his life on problems of convexity and on the geometry of numbers? Obviously he refers to euclidean geometry of the type taught in high schools of his period.

Minkowski had a remarkable geometric intuition which he brought to bear on problems of number theory among other things. His theories on pencils of convex bodies were instrumental in the creation of integral geometry(42) and the new trend in the geometry of numbers.(43) The arithmetization of geometry which Hilbert tried to establish(44) did not overly excite Minkowski as Minkowski's intuitive geometric ideas stirred up Hilbert's desire to get at the truth without geometric intuition.(45)

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(38) s. Hurwitz [3].

(39) s. Minkowski [14], [15].

(40) Minkowski's letter speaks about a good deal more than is contained in [11]–[13].

(41) The book did not get finished in the intended form, planned probably as the second volume of [14]. Some of the material went into [15] which if properly studied, provides one of the best introductions into algebraic number theory.

(42) s. Blaschke.

(43) s. R. L. Graham, Witenhausen, Zassenhaus.

(44) in the "Foundations of Geometry."

However Minkowski also saw what Hilbert did not see, that both geometric intuition and conceptual brilliance cannot stand in the 'user's court' if they are not underpinned by a broad algorithmic foundation. He tried in particular to establish algorithms for obtaining the units of an algebraic number field,(46) though he also very probably was convinced of the numerical inadequacy of his estimate for the class number of a field in terms of its discriminant. In other words he tried hard to establish the foundations of constructive algebraic number theory, a goal we begin to envisage anew now many years after Minkowski.(47) Note how little Minkowski is impressed by the pedagogical concerns of a text book writer!

IV. On Applied and Pure Mathematics

December 29, 1887 (from Bonn): My colleague v. Lilienthal is a very charming man; but I like to talk with him on anything else but mathematics. Quite soon he gets much too deep and raises questions on concepts and foundations when I want definite facts.

December 22, 1890: That I do not come to Königsberg at Christmas, you will have gathered already from the fact that I did not turn up, neither in your study nor at the certain Nachtigallensteigecke, nor anywhere else on the horizon. I do not know whether I must console you, nor even do I know whether it gives you comfort to hear from me that this time you would have had little joy from me as someone who is thoroughly infected by physics. Perhaps I would have to undergo a ten day quarantine before you would have readmitted me to our joint walks as mathematically pure and unapplied.

In order to find things in common with other mortals I have yielded to magics (meaning: to physics) entirely. I have practical laboratory sessions in the physics institute, at home I am studying Thompson, Helmholtz and consorts; and beginning with the end of next week I shall even work in a blue smock a few days a week in the institute for the construction of physical instruments, a type of practical activity more terrible than anything you can imagine.

April 16, 1895: . . . I miss your company very much. I didn't quite make it yet to the apple tree, I believe, since your departure though I walked frequently with Garbe who always depicts the charms of the local professorial daughters.

July 1, 1895: The actual infinite is a term that I came across in a paper of Cantor, and I referred in my lecture(48) mostly to theorems of Cantor too; they found general interest though some people did not want to believe in them. The actual infinite in nature of which I mainly spoke—the title was worded somewhat excitingly so as to attract a few people inspite of the heat—was the position of points in space. Whether the attribute "in nature" was justified or not, I must leave undecided, even though I had constructed some amusing examples for Cantor's theorems in that regard. At this opportunity I have realized once again that Cantor is one of the most inspired living mathematicians.

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(45) s. Minkowski [15] Kap. I §6–9, s. also Hecke, 116–117.

(46) s. Minkowski [13].

(47) s. Zimmer.

(48) s. Minkowski [16].

February 11, 1899: Also I am still much occupied with the applications. From thermodynamics I went to chemistry. I always think one of these days to come to the rescue of (F.) Klein against his many attackers by way of demonstrating that the mathematicians really can do something for the practice, and even better things than to record the movements of a spinning top.

Comments: We observe that Minkowski is a 'naive' mathematician from the viewpoint of a contemporary having gone through all the strictures of the 'foundations of mathematics'. Nevertheless he develops subtly the consequences of Cantor's set theory in a physical interpretation leaving it open whether perhaps a discrete interpretation would be more appropriate. The perpetual discussion by word and example between Minkowski, F. Klein, and D. Hilbert on 'pure and applied mathematics' is still with us, even in about the same terms and so far with about the same results too.

F. Klein(49) in numerous books and articles tried to formalize the applications of mathematical thought as an independent discipline of teaching and research. Though this approach is most popular among us its dangers are a certain lack of depth and a lack of interest of the pure mathematicians. Hilbert, on the contrary, tried to purify the applications with his theory of gas equations(50) and general relativity.(51)

The dangers of this approach are its lack of applicability and the lack of interest it found with physicists in the precomputer age.

On the other hand, by the purely mathematical theory of integral equations, Hilbert and his pupils (like E. Schmidt and H. Bückner) perfected both the conceptual and the numerical aspects of Fredholm's theory so as to anticipate the need of a most useful tool for the modern quantum physicist.(52)

Minkowski, however, tries to listen to his colleagues, the physicists and chemists, hoping that what they tell him may strike a sympathetic chord in his own mind. In a manner of speaking he believes in team work. His hopes, as expressed in his letter of February 11, 1899, found their fulfilment in the last years of his life when he returned the stimulation he received from his friends, the physicists, with interest in the form of classical treatises on special relativity theory(53) and electromagnetic theory.(54)

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(49) s. e.g. Klein und Sommerfeld. For a critical appreciation of Klein's approach, s. Heun, in particular the summary at the end of the article (p. 117–120).

(50) s. Hilbert [9].

(51) s. Hilbert [10]–[11].

(52) s. C. Reid 171.

V. On Hilbert's Paris Address on "The Problems of Mathematics"(55)

December 30, 1899: Regarding your plans for a speech in Paris I cannot give any opinion yet, I shall think it over still and soon write about it to you.

January 5, 1900: I studied again Poincaré's Zürich address.(56) One can well subscribe to all his assertions in view of the mild way they are stated. Indeed he does full justice to pure mathematics. Therefore a speech exclusively in praise of pure mathematics does not fully convince me as the best thing to do. By the way only a few people will still remember what Poincaré said at that time. Since Poincaré was not there himself at the time and his draft was read, the impression was not nearly such as with Boltzmann at Munich for example.(57)

Most attractive would be the attempt of a preview on the future, in other words the determination of the problems, which future mathematicians should attack. Here your could achieve that people even after many years would still speak of your address. Yet to prophesy is of course a difficult thing to do. Perhaps you may be also afraid of revealing prematurely certain ideas which you have formed regarding the future treatment of problems. Topics of a more philosophical nature are perhaps better for a German public than for an international one. A review and a preview will be probably also given by a French mathematician who is likely to be the first to speak. You should try somehow to make sure about that. Since the audience are experts after all I think an address like Hurwitz's which also was liked very much at the time,(58) with definite facts would be more appropriate than a mere chat like Poincaré's.

I am really keen to know on which topic you will finally settle. Though it is much more the exposition than the topic itself which matters, nevertheless one could effect a doubling of the audience by an appropriate choice of a theme.

July 10, 1900: Of the more definite (?) problems of mathematical physics which are neither of a too special nor a too general character one should perhaps mention to find mechanical analogues of the mode of action of the forces in ether and furthermore the forces of chemical bonding . . . I believe that there are really many mathematical questions that are interesting and useful for physics in the areas touched by Boltzmann (Gastheorie, Bd I §1) Einleitung, S.4.5. Bd. II Schluss des §70. S. 206, S. 212 Mitte der Seite, §88–90.) Perhaps you also ask Nernst for his opinion. Also with the philosophers you will have to fight some tough battles.

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(53) s. Minkowski [18].

(54) s. Minkowski [17].

(55) s. Hilbert [12].

(56) s. Poincaré [13].

(57) s. Boltzmann.

(58) s. Hurwitz [2].

At any rate, that is both Hurwitz's and my own opinion, you must shorten and cut the draft very substantially for oral presentation. For this will be better since some passages have to be spoken at a slower speed and you don't have to use all of your time. I shall read of course with great interest the galley proofs of your address.

July 17, 1900: I have received the first 3 proofs of your address. Since I don't know what is still to come I cannot yet give my definite opinion. At any rate it is extremely original to pose as a problem for the future that which mathematicians believed to possess already for the longest time, as the axioms of arithmetics. What will be the reaction of the laymen who will be represented among the audience in great numbers in any case?

July 28, 1900: I have finished reading your address with great pleasure. Since I waited for the end to get the correct picture my reading was somewhat delayed. I can only congratulate you on the address, it will certainly be the event of the congress and the success will be long lasting. In particular I believe that your power of attracting young mathematicians through this address which probably will be read by every mathematician without exception will still grow if this is possible at all. By elimination of small things the introduction has gained in perfection. Hurwitz and I had made ourselves an entirely wrong picture thinking that there should be called a halt to the 'Ignorabimus',(59) in particular since you had dealt with the variational calculus in such detail already.

Now you really have taken mathematics under your command and one will generally acknowledge you to be the director general. Now I am keen to know how you will proceed with the oral presentation. You can't say everything. Perhaps you read right on to the Ignorabimus, you give afterwards the 21 problems in the sectional session, referring to them in the address and still read the terminal passage. But, this procedure may not work in the end, if your address will be placed on the agenda of Friday rather than Monday.

Comment: The story of the preparation of the Paris address has been told very well already in C. Reid's book. There are undertones supplied by Hilbert's and Minkowski's relation to the French mathematicians as well as to German mathematicians like Dedekind and Frobenius.(60)

It is remarkable how well the design of the completed address fits the general pattern indicated by Minkowski.(61)

And yet Hilbert's address almost has gone its full course in the history of our science so that even a large meeting of a predominantly historical bent relating to the Paris address can be held nowadays in the USA. But Minkowski's ideas are still as fresh as when they were first born and their full impact has hardly been explored. Thus we see in this country many young mathematicians of the most varied interests working on problems and areas of problems with which Minkowski wrestled already.

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(59) s. Hilbert [12] 298 (2nd paragraph). Emil Dubois-Reymond (1818–1896), a German physiologist had stated in a famous address on the laws of nature, that we will never know the internal workings of nature: Ignorabimus. Hilbert answers: "In der Mathematik gibt es kein Ignorabimus."

(60) s. C. Reid 86, Biermann

(61) s. C. Reid 69–73.

There are the number geometers all stout of heart and fearless. There are the functional analysts forgetful of Minkowski's more geometric ideas. There are the functional equationists taking up on a grand scale Minkowski-Hilbert's algorithmic ideas. There are combinatorialists, especially of the geometric type conjuring up the most sophisticated space divisions which would give Minkowski great joy to behold. There are applied mathematicians of all descriptions some working in teams, some working alone, some listening, some talking. There are number theorists of all denominations dealing with prime numbers and discriminants, so as to make them 'wiggle and waggle',(62) and we begin to out-dream Kronecker. There are the convex and also the non-convex geometers and there are the analysts sometimes painting doors and varnishing windows, sometimes engaging in sophisticated speculations of a topological nature which in principle were quite familiar to Minkowski as his recently rediscovered lecture notes show. And last but not least the algebraists who nowadays do a great deal more than Minkowski and Hilbert would dream of when they did not study Dedekind.

Here is a young generation full of vigor, apparently much less afraid of any lack of professional success than Minkowski and Hilbert were in their younger days, on the contrary we see them taking over gleefully one position after the other. May our science continue to prosper with them and after them!

Let me finish with a number of questions of continuing interest.(63)

1. Give algorithms that are best possible on the average for computing class number, unit group, discriminant and maximal order of any Dedekind order, and the Galois group of an algebraic number field.(64)

2. Classify algebraic number fields in terms of the behavior of its ramified prime numbers.(65)

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(62) In Fall 1893, Hilbert an Extraordinarius (associate professor) of the University of Königsberg, was offered by F. Althoff the position of an Ordinarius (full professor) at the same University vacated by Hilbert's teacher Lindemann. Hilbert, when asked by Althoff to name a successor for his former position, succeeded, in spite of some difficulties, in bringing Minkowski to Königsberg for Easter 1894. When Minkowski had accepted Althoff's offer he wrote to Hilbert on January 3, 1894 ([1], 59): Hearty thanks for all your efforts which lead to this happy end; let us hope for an agreeable and fruitful time together so that the prime-numbers and reciprocity laws may wiggle and waggle!

The two friends stayed jointly as colleagues at Königsberg until Easter 1895 when Hilbert moved to Göttingen.

(63) Growing out of the intimate acquaintance with Minkowski's work.

(64) s. Zimmer, Zassenhaus [1], [2].

(65) s. Zassenhaus [3].

3. Study the optimal packing and covering inequalities in three and more dimensions.(66) Is it true that in case of the existence of a Lie group G for a packing or covering problem there are optimal distributions with an automorph contained as a discrete subgroup with compact factor space in G?

4. Apply higher arithmetics (reciprocity laws) effectively to the problem of prime factorization.(67)

5. Establish algebraic model theory as a theorem proving device.

6. Establish fully the relationship between the integral representations of a Dedekind order (e.g. the integral group ring of a finite group) and its representation theory modulo prime ideal powers.

7. If there are discrete space time universes, which mathematical features are in common to them to make them compatible with the macroscopic behavior of the observable world?

8. Will this country continue to encourage the growth and maturing of the apparently diverging research tendencies of its mathematicians or will it succeed in effectively manipulating them?

9. Do we work on a problem because it hounds us or because someone else holds us bound to it?

10. What is more important: to clarify a concept in our mind or to publish many papers?

11. Can we afford to continue proving the existence of mathematical objects which we cannot construct?

12. Does applied mathematics mean to profit from what we can do well, or to do routinely what once some people could do extremely well or to listen responsively to the questions of others?

Résumé

L'histoire des sciences mathématiques est beaucoup plus que l'étude de documents et la vérification de dates importantes.

Les documents et les dates sont nécessaires pour rappeler a ceux qui étudieront plus tard la profondeur véritable des mathématiques, quelles furent les étapes cruciales du passé.

Il est vrai cependant que chaque nouvelle génération de chercheurs va nécessairement réinterpreter l'essence et l'impact de découvertes passées à la lumière de leur propre perception. Ceci dans le but de regrouper le peu qui soit connu et d'ouvrir la voie à des nouvelles découvertes.

Prenons donc un moment de recul hors de nos sphères actuelles de préoccupation pour nous replonger dans notre passé commun et retrouver encore une fois une vision future de ce qui s'ouvre devant nous.

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(66) s. Graham, Witsenhausen and Zassenhaus.

(67) So far only elementary number theoretic and quadratic reciprocity law methods have been applied to the problem of effective prime factorization.

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BOOK REVIEWS—COMPTES RENDUS CRITIQUES

Model theory for infinitary logic, by H. J. Keisler. x+208 pages. North-Holland, Amsterdam, 1971. U.S. \$11.00.

This is the first book to be published on infinitary logic and is concerned with the logic $L_{\omega_1\omega}$. This logic is the extension of first order predicate logic L obtained by allowing countably infinite disjunctions and conjunctions. The book is designed for those who already have a good knowledge of first order logic. Being based on a lecture course of the author it should prove an excellent text for an advanced graduate course. It is indispensable for those doing research in mathematical logic.

In the early part of the book much use is made of consistency properties in the style of Makkai to prove some of the basic results on $L_{\omega_1\omega}$. The method of consistency properties is an elaboration of the method by which Henkin proved Godel's completeness theorem for L. The resulting proofs are short and easy to understand. Many of the routine parts of proofs are left to the reader. This makes for conciseness and focuses the reader's mind on the essential steps of each proof.

Just as the method of consistency properties has a counterpart in the theory of L so do most of the methods available in the study of $L_{\omega_1\omega}$. One cannot help being struck by how close the methodology of $L_{\omega_1\omega}$ is to that of L. The similarities and contrasts between L and $L_{\omega_1\omega}$ are noted throughout the text.

This is an excellent addition to the literature for specialists in mathematical logic. Non-specialists should find it useful as a reference book provided they know the elements of first order logic.

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Temporal Logic, by Nicholas Rescher and Alasdair Urquhart. (Library of Exact Philosophy, Vol. 3). xviii+273 pages. Springer Verlag, New York—Wien, 1971. U.S. \$19.80.

This is a detailed study of how propositional logic can be extended so as to permit the treatment of statements depending on points in time. To start with, take a usual propositional language the formulae of which are construed to be absolute, i.e. time independent statements; take a set D of time symbols t and introduce, for every old formula A, a new formula $R_t(A)$ with the intended meaning that A holds at t; finally, form more new formulae by closing up under propositional connectives, under the operators R_t with t in D, and under universal quantification $\forall t$. The intended meaning then makes it plausible to introduce axioms stating that each operator R_t commutes with the propositional connectives as well as with quantification. As for the interpretation of iterated operators, say in a formula $R_t(R_{t'}(A))$, there are two possible attitudes. The one view takes $R_{t'}(A)$ to be temporally definite such that nothing new comes from prefixing an additional R_t; thus $R_t(R_{t'}(A))$ is postulated as equivalent to $R_{t'}(A)$. The other view takes D to be a group—preferably one of real numbers—and $R_t(R_{t'}(A))$ is postulated as equivalent to $R_{t+t'}(A)$. In both cases, the axioms stated show that the operators R_t can be pushed in front of the propositional variables X in such a way that every formula becomes equivalent to a normal form built from formulae $R_t(X)$ with help of propositional connectives and quantification. Viewing $R_t(X)$ to be a monadic predicate $X(t)$, one arrives at a monadic predicate calculus; in the first case where there are no stipulations concerning D, this is the full monadic predicate calculus, while in the second case it is a fragment depending on the structure which was put upon D.

Now the entire situation can be generalized in an obvious way by allowing atomic formulae $t=t'$, for t,t' in D, in addition to the propositional variables X; one thus obtains monadic predicate calculi with equality. Consider now the case of an unstructured D. Although the formal apparatus allows for the quantification over time, the temporally definite interpretation of formulae $R_t(A)$ suggests that the t here has more the character of a constant than that of a variable. There are, however, syntactical as well as semantical situations which make it desirable to have available a time index having the properties of a proper variable; thus D is enlarged by adding a symbol n, obtaining a set D', and the set of formulae is enlarged by now permitting the t to come from D'. A formula $R_n(A)$ then is temporally indefinite, taking the meaning "A holds presently" where presently refers to a shifting instant of time (in contrast to the usual, definite hic-et-nunc meaning of "now"); thus A shall always be equivalent to $R_n(A)$. This n now permits substitution: $R_t(n=t')$ is equivalent to $t=t'$, $R_t(t'=n)$ is equivalent to $t'=t$; with respect to quantification, however, n is a non-entity, and the rules of the predicate calculus cannot be applied to formulae containing n. In case D carries the structure of a group, the rôle of n can be taken by the zero element although the substitution axioms still have to be added explicitly. Assume now that D' is provided with a binary relation $<$ which, in case of D carrying a group structure, shall be compatible with the group operation. One then can define the four tense operators F, P, G, H (future, past, future-perpetual and past-perpetual): $G(A)$ is $\forall t(n<t\rightarrow R_t(A))$, $H(A)$ is $\forall t(t<n\rightarrow R_t(A))$, F is $\neg G\neg$, P is $\neg H\neg$; combinations of these operators permit the expression of more complicated tenses. On the other hand, one may construct a tense language, built from the propositional language by permitting the formation of formulae $F(A)$, $P(A)$, $G(A)$, $H(A)$ with help of new symbols F, P, G, H; it then is obvious that the above definitions give rise to a syntactical transformation, mapping the new tense language into the earlier time language. The first surprising result now says that there is an axiomatization, formulated within

the tense language (and due to E. J. Lemmon) such that the theorems of that axiomatization are precisely those formulae whose translations are theorems with respect to the axioms for the time language mentioned above. The proof given here is semantical, making use of (a) the fact that tense language can be interpreted in structures permitting also the interpretation of time language, and (b) a completeness theorem for the axiomatization of tense logic with respect to these structures; the latter is proved via semantic tableaux generalizing those employed by Kripke for modal logic. There now follows a number of investigations, establishing axiomatizations in tense logic of those fragments of time logic which arise if $<$ is assumed to have the respective properties of being transitive, a tree branching into future, a linear order (N. B. Cocchiarella), a linear order without first or last element (Dana Scott), a dense linear order (A. N. Prior), a continuous linear order (R. A. Bull) or a discrete linear order (R. A. Bull); in each case a completeness theorem with respect to the appropriate tense structures holds. The mathematical reason behind these logical theorems lies in the fact that, in a first order language with predicate symbols $=$ and $<$, various order theoretic properties of $<$ can be expressed by formulae involving the predicate operators G and H, applied to arbitrary monadic predicates A; here G is defined by $GA(x): \forall t(x<t \rightarrow A)$ and H by $HA(x): \forall t(t<x \rightarrow A)$, t being free in A. For instance, if $<$ is transitive then $GA(x) \rightarrow GGA(x)$ holds; conversely, if this formula holds and if $a<b<c$, define $A(t)$ to be $a<t$ whence $GA(a)$ implies $GGA(a)$; thus $a<b$ implies $GA(b)$ and, therefore, $b<c$ implies $A(c)$, $a<c$.—The next topic to be taken up then is the temporal interpretation of modality. There are two possible views here: the Diodorean defining necessity by $\Box A$: $A \wedge G(A)$, possibility by $\Diamond A$: $A \vee F(A)$, and the Aristotelian-Megarian defining necessity by $\Box A$: $A \wedge G(A) \wedge H(A)$, possibility by $\Diamond A$: $A \vee F(A) \vee P(A)$. Depending on the order properties of the time relation $<$, one then obtains various known modal systems: for arbitrary $<$ the Diodorean definition gives the Von Wright-Feys system **M**, and the Aristotelian definition gives the Brouwerian system **B**; in case that $<$ is a tree branching into future the Diodorean definition gives **S4**, and in case that $<$ is transitive only the Diodorean definition gives again **B**; if $<$ is linear and not discrete the Diodorean definition gives **S4.3** while in the discrete case a stronger system D results; the Aristotelian definition gives **S5** for every linear $<$.

The remaining chapters of the book deal with items more of interest for the philosopher: further possibilities for modalities; temporally conditioned descriptions and different conceptions of time; processes; the logic of world states and a world history; the master argument of Diodorus Cronus with a discussion of temporal determinism, future contingency and a possible 3-valued conception of temporal truth. The book is quite readable for a mathematician with some background in mathematical logic; its presentation is immensely superior to Prior's *Past, Present and Future* (Oxford U.P. 1967) which is often referred to for philosophical matters. For possible future editions this reviewer would recommend a more extensive treatment of first order temporal logic than the one presently given in chapter XX; this could take into account, among other things, Grzegorczyk's logic of scientific research (Indagationes Math. **26** (1964) 569–601), cf. also Klemke's work (Archiv f. Math. Logik u. Grundlagenforsch. **14** (1971) 148–161) and the thesis of Görnemann (Hannover 1968). Also, a discussion of the constructivist uses of time for defining negation and quantification, as employed in Kripke's semantics for intuitionistic logic and put to use in the forcing technique of set theory, would fit neatly into the general framework of temporal logic.

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Linear Associative Algebras, by Alexander Abian. vii+166 pages. Pergamon Press, New York, Toronto, etc., 1971. U.S. $11.00.

It seems that before the appearance of this book no modern introductory work in the English language existed on this topic. Thus the author has filled a substantial gap in the literature. The book should enable a student after a first course in abstract and linear algebra, without elaborate knowledge of abstract ring theory, to learn the basic facts on finite-dimensional algebras. The book is skilfully written and carefully directed toward a reader who does not yet know the subject. The preparations in the first two chapters are ample for a smooth development of the main theme in the last two chapters. Occasional repetitions and side remarks make the reading quite pleasant. Every section contains a set of appropriate exercises (without answers), all non-trivial and some of them hard, which should help to extend the reader's grasp of the theory.

Chap. 1, "Preliminaries" (p. 1–38) reports on algebraic systems in a general way and deals in some detail with the direct sum and the tensor product of rings, with polynomial- and matrix-rings. Chap. 2, "Vector Spaces" (p. 39–64) presents the fundamentals of Linear Algebra and Matrix Theory. Chap. III, "Linear Associative Algebras" (p. 65–121) introduces the main concepts of the theory: Idempotent and nilpotent elements, ideals, simple algebras, quotient (difference) algebras. After a section on canonical forms of matrices over a field follows a section on matrix representation of an algebra and a section on division algebras, ending with a proof of the Frobenius Theorem. Chap. 4, "Wedderburn's Structure Theorems" (p. 123–157) applies the ring operations, described in Chap. 1, to algebras, based on the representation of an algebra by matrices. The concept of the radical of an algebra and that of a semi-simple algebra is introduced. The main theorem, stating that every semi-simple algebra is the direct sum of simple algebras is proved. The last section demonstrates the various characterizations of simple algebras.

The last chapter (4) appears to be incomplete due to the absence of a proof of

Wedderburn's "Principal Theorem" according to which every algebra splits directly in a semi-simple algebra and the radical. The theorem is, indeed, stated in the "Concluding Remarks"; but the hint to a proof, consisting of four lines of references to earlier theorems, is probably useless. Of course a reader who has followed the author to this point can now proceed with A. A. Albert's "Structure of Algebras".

Further it would have been interesting to see the theory "at work". As far as semi-simple algebras are concerned, a more detailed theory of group algebras might have been expected, followed by some theorems on group representations.

Finally a word on the Bibliography. It is restricted entirely to books in the English language, mainly on abstract and linear algebra. Of those specifically on (hypercomplex) Algebras only Albert's Colloquium Publication (1939, see above) and L. E. Dickson's "Algebras and their Arithmetic" (1923, unchanged reprint 1960) are quoted, in addition to the somewhat archaic paper by B. Peirce "Linear Associative Algebras" (Amer. J. Math. 4 (1881)) which would be of interest only from the historical point of view. In this respect—and otherwise—Dickson's earlier Cambridge Tract "Linear Algebras" (1914) also could have been mentioned. For the serious student of algebra theory, however, it will be quite impossible to avoid the Ergebnisse Volume "Algebren" by M. Deuring (1935, re-edited 1948) and, again mainly from the historical point of view, the articles by E. Study "Theorie der gemeinen und hoeheren komplexen Groessen" (1898) and by E. Cartan "Nombres complexes" (1908) in the German and French Mathematical Encyclopedia respectively. Also of independent interest is Dickson's "Algebren und ihre Zahlentheorie" (1927), the extended translation of his book mentioned above. Finally there is N. G. Čebotarëv's "Introduction to the theory of Algebras" (1949, in Russian) which, with regard to its aims and certain of its contents, is similar to this book by Abian.

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Multivariate Analysis, by A. M. Kshirsagar. (Statistics Ser. Vol. 2). xiv 534 pages. Marcel Dekker, New York, 1972. U.S. $19.50.

Traditional one year advanced graduate level courses on multivariate statistics usually comprise a discussion of testing and estimation and the related distribution problems (central case) for the general multivariate normal situation—discriminant analysis is also viewed as a testing problem. Kshirsagar's new book has no trouble in meeting this need. Using elegant matrix methods the book achieves this goal in a clear, concise and comprehensive manner. Random orthogonal transformations and Bartlett's decomposition (via triangular matrices) of the Wishart matrix have been well employed in solving the standard distribution problems. This makes the presentation lucid, consistent and, for a work of this sort, strangely enough, readable. To make the book not only a reference source for the graduate student but also an advantage to the modern researcher many of the recent results which fit in this framework have also been included. For example, results on optimality have been mentioned and references given, although a fuller discussion of these results, at least in the simpler cases, would have been appropriate.

Some criticism of the shortcomings of the book are also in order. Notation could have been simpler—more traditional. For example $\sqrt{-1}$ was not found to be an advantage over i and $n!/n_1! \ldots, n_m!(n-n_0)!$ is a much harder thing to look at than $(n/n_1, \ldots, n_m, (n-n_0))$. Notation throughout the book was found to be quite burdensome and the reviewer gets the impression that not much attention was paid to this important aspect of presentation. The discussion of multiple correlation and regression would have been more readable if the author had chosen the variables $y; x_1, \ldots, x_p$ rather than $x_p; x_1, \ldots, x_{p-1}$. The number of misprints is enormous and the errata at the end of the book is even in error (see e.g. correction for page 122). There are many vague statements besides the inaccuracies. For example in the reviewers opinion not enough evidence is available yet to make the statement on page 3, "...and may times these techniques are quite robust." On page 10 an equation at the bottom of the page does not define a "regression line". On page 12, the definition of the multiple correlation coefficient is incorrect. It is not the positive value of the square rest of (3.8). The multiple correlation is by definition a positive quantity. On page 29, equation (2.5) what he is really using is $\mathrm{tr}AB = \mathrm{tr}BA$ but one gets a different (more complicated) impression. On page 30, it is impossible to choose all the elements of the columns of C positive. One can of course take all diagonal elements positive. On page 51, it is not correct to say that, "In the next section, we shall consider the more general case when the variance-convariance matrix of the multivariate normal population is not I, but Σ, and $\boldsymbol{\mu} \neq \mathbf{0}$." This gives the impression that he is going to derive the non-central distribution. He does not. On page 74, while (6.3) is correct, the explanation given in the square bracket is not correct. Actually the solution involves complex integration. On page 78, the noncentral Wishart distribution depends on the matrix $\Sigma^{-1}MM^1$, not just on its roots. Although most of the chapters have been written in a balanced way, the reviewer found the discussion of optimallity results in chapter 5, section 6, to be incomplete. Especially the important results by Linnik, on minimax, are missing. Also, in chapter 6, the results of John on the missclassification of errors are not even mentioned; admissibility results in the unequal convariance matrices case are mentioned but glossed over, and no reference is even given in the equal convariance case. Okamoto's Expression (6.4) page 220, is meaningless without an explanation of the constants involved *and* even that authors original paper has a correction note which has not been mentioned. It is a pity that some of the easily derivable matrix results are not presented in an Appendix rather than referred to the original source, This is particularly unpleasant in cases where the reference is fairly intractable itself for example Theorem 2 on page 260.

M. S. Srivastava,
D. Brenner

Problems in Complex Variable Theory, by Jan G. Krzyz. xvii+283 pages. American Elsevier Publishing Co., Inc., New York; PWN—Polish Scientific Publishers, Warszawa, 1972. U.S. $18.50.

This is a collection of problems (with solutions) in a single complex variable. The problems are for the most part classical, at the level of senior honors/first year graduate study. It should prove useful as a supplementary reference source, both for student and instructor.

J. L. Fields,
University of Alberta

Combinatorial Methods, by J. K. Percus. ix+194 pages. Applied Mathematical Sciences 4, Springer-Verlag, New York, 1971. U.S. $6.50.

Combinatorial Methods is the outgrowth of a course in combinatorial methods given at the Courant Institute. The book makes heavy demands on the student's preparation. For example, he is expected to be familiar with residue calculus, Fourier transforms, and the Perron-Frobenius Theorem. A number of examples are given which convince the reader that there are useful nontrivial applications of combinatorics in physics and chemistry.

The book is in two chapters. Chapter I, Counting and Enumeration on a Set, includes generating functions, the Master Theorem on permutations with restricted position, and Ramsey's Theorem. Chapter II, Counting and Enumeration on a Regular Lattice, includes Random Walk on lattices, the Dimer problem, the Ice problem, and the Ising Model.

The mathematical exposition is very clear. However theorems and definitions are seldom displayed. Sometimes one has to reread a section to decide just what is the theorem that entitles the section.

This is an impressive book. There is a lot of material presented in less than two hundred pages. The style is terse, but convincing.

Nora Losey,
University of Manitoba

Uniqueness Theorems in Linear Elasticity, by R. J. Knops and L. E. Payne. ix+130 pages. Springer Tracts in Natural Philosophy Vol. 19. Springer-Verlag Berlin Heidelberg 1971. U.S. $10.40.

The Tract under review is mainly concerned with weakening the hypotheses on the elasticities $c_{ijk\ell}(\underset{\sim}{x})$ in discussing the classical uniqueness theorem for the equation

$$\frac{\partial}{\partial x_k}\left(c_{ijk\ell}\frac{\partial u_k}{\partial x_l}\right)+\rho F_i=\rho\,\frac{\partial^2 u_i}{\partial t^2}$$

under initial-boundary conditions, with existence assumed. Throughout the Tract stresses and displacements are assumed to be continuous. The definition of weak solutions includes among the usual generalizations, a conservation law.

The Tract is mostly self contained, although a reader who is familiar with the classical theory of elasticity would, naturally, be better motivated. The book provides the reader with all that he needs to tackle modern problems regarding uniqueness theorems in linear elasticity, although they may need a discussion of the regularity of the stresses and displacements. Uniqueness theorems for plates and shells or for deformation of rods or beams are not considered.

The historical development of the topic is traced in Chapter 1 which ends with a list of problems yet unsolved. This is followed by a description of the basic equations in Chapter 2 and an account of the early work in Chapter 3. In Chapter 4, anisotropic and isotropic considerations for bounded regions, exterior domains, traction boundary value problems and mixed boundary value problems are considered for three dimensional elastostatics. Isotropic two dimensional elastostatics form the contents of Chapter 5 while problems in the whole- and half-space are discussed in Chapter 6. Chapter 7 discusses the sphere problem, Cauchy problem and the Signorini problem involving ambiguous conditions. Perhaps the most important chapter, Chapter 8, discusses uniqueness theorems in elastodynamics and collects some results connecting the concepts of stability, boundedness, existence and uniqueness.

The authors have given an eminently readable account of the topic and in several areas have been principal contributors. The Tract will surely become a motivating force among theoretical elasticians and those who are involved with theory of partial differential equations.

P. N. SHIVAKUMAR,
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Lectures on Celestial Mechanics, by C. L. Siegel and J. K. Moser (Grundlehren der mathematischen Wissenschaften, Band 187). xii+290 pages. Springer Verlag, New York, Heidelberg, Berlin, 1971. U.S. $22.60.

"Lectures on Celestial Mechanics" is not a mere translation of "Vorlesungen über Himmelsmechanik" (Grundlehren der Mathematischen Wissenschaften, Band 85) which appeared in 1956, but is a completely revised and considerably enlarged version of the now famous book. The present version contains two new sections on triple collision in the three-body problem, and a new and simpler proof of Siegel's theorem on conformal mappings near a fixed point. The contributions of the second author on stability theorems for systems of two degrees of freedom and existence theorem for quasi-periodic solutions are included in the latter sections of the book. The addition of this new material has made this book even more valuable, even more important.

For those who are not familiar with the subject matter, it is probably appropriate to say that the book deals with the so-called n-body problem in Classical Mechanics, which in Siegel's words ". . . consists of describing the complete behaviour of all solutions to . . . equations of motion (of n bodies) for arbitrary pre-assigned initial conditions. Despite efforts by outstanding mathematicians for over 200 years, the problem for $n>2$ remains unsolved to this day . . . At the suggestions of Mittag-Leffler, the King of Sweden and Norway established a prize for the solution to the problem of finding a series expansion for the coordinates of the n bodies valid for all time." This problem was solved by Sundman for $n=3$. An application of Sundman's results to the earth-sun-moon system leads Siegel to conclude that "Under the (above) assumptions, a direct computation will then give two positive numbers ρ and ε such that the moon, at a potential collision of the earth and sun, must be at least a distance ρ from the earth and require at least an additional time ε before possibly colliding with it. Thus, Sundman's theory allows us to view the future with confidence."

The book contains a wealth of original ideas, useful tools and valuable insights, and one would have no hesitation in placing it among the great books of Poincaré, Birkoff and others on the subject.

D. K. SEN,
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Problèmes de Topologie, par Jacques Chaillou et Jacques Henry. 101 pages. Masson & Cie., 1971.

Le titre complet de ce petit livret donne plus de précisions sur con contenu: Problèmes de topologie (avec des solutions détaillées) extraits du cours d'analyse, Tome II, Topologie, de M. Choquet.

Les auteurs ont choisi 22 problèmes sur plus de 110 pour le chapitre V (Espaces topologiques et espaces métriques), 14 sur plus de 50 pour le chapitre VI (Fonctions numériques), 27 sur plus de 200 pour le chapitre VII (Espaces vectoriels topologiques). C'est dire que bien des problèmes du cours de M. Choquet ont été

laissés de côté. Cependant, le choix a été fait de telle sorte que si un problème exige la solution d'un autre problème, cet autre problème a été aussi résolu. Tel quel, ce travail peut procurer une aide efficace à l'étudiant qui travaille seul. D'abord il y trouvera la solution de problèmes rédigée d'une façon très soignée et très claire, chaque "geste" mathématique étant justifié par une référence très précise au résultat du cours utilisé. Ainsi il prendra "confiance pour aborder d'autres problèmes". De plus, quelques commentaires tendent à élargir l'horizon, à prendre une meilleure conscience des rapports de la question traitée avec d'autres semblables, ou à donner d'autres directions de solutions, . . . Il arrive même qu'un énoncé se révèle incorrect au cours de la solution, et le commentaire le rétablit dans toute sa rigueur (cf. exercice no. 129, commentaire page 80).

De très rares fautes typographiques ont été relevées, pour la plupart sans aucun danger. A noter cependant une exception: de regrettables confusions d'indices et d'exposants s'introduisent à la fin de la solution de l'exercice no. 143, qui ne manqueront pas de troubler les lecteurs auxquels ce genre de texte s'adresse.

Une dernière remarque à l'adresse du public mathématique anglophone. La partie du cours d'analyse de M. Choquet dont ces problèmes de topologie sont extraits est l'original français du livre intitulé Topology publié par Academic Press en 1966 (# 19 in series of Monographs and textbooks in Pure and Applied mathematics). Les références (nombreuses) se rapportent aussi bien au texte anglais si on fait abstraction de la pagination (bien que même là, l'approximation soit très bonne, les valeurs indiquées étant toujours par défaut, avec une erreur d'une page au début, et au plus de dix pages à la fin).

Donc une oeuvre utile à l'usage des étudiants du niveau de la Maîtrise qui désirent s'initier à l'analyse moderne, et qui ont déjà apprécié le Cours de M. Choquet, ou qui veulent le découvrir.

Introduction to Boolean Algebras, by Ph. Dwinger. 2nd, revised and enlarged edition. v+71 pages. Physics-Verlag, Wurzburg, 1971.

This new edition succeeds, as did its predecessor, in dealing with a large amount of material in comparatively few pages. Besides covering the basic 'universal algebra' of boolean algebras (homomorphisms, free algebras, etc.) and discussing notions such as atomicity, completeness, and measures, it is an excellent reference on representations of boolean algebras by fields of sets and on the connections between boolean algebra and topology. The topics added in the new edition: free α-complete boolean algebras, superatomic boolean algebras, and free α-complete and free complete extensions, are both interesting and useful and fit in well with the main body of the text. A possible criticism is that, whilst there are references to the further literature on the subject, these could be supplemented, particularly in regard to the more recent literature: for instance to the work of Hanf and of Pierce on the classification of countable boolean algebras.

Perhaps more important for an introductory book is that the relations of the subject matter to other branches of mathematics be mentioned. The book under review is certainly adequate with regard to boolean algebras and topology, and measure theory is mentioned, though perhaps fuller references might have been given to the occurrence of boolean algebras in analysis (in spectral theory, for example). The fact that there exist intimate connections between boolean algebras and logic could be more in evidence; this was the original motivation of the subject and there has continued to be a fruitful interplay (for just one instance, boolean-valued models of set theory have not only been valuable in foundational work but have also enabled some questions on complete boolean algebras to be resolved). But a monograph is not a treatise with applications, and the present book continues to be a worthwhile item in the boolean algebra literature.

D. A. HIGGS,
UNIVERSITY OF GUYANA

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